Supernova Modeling via Kinetic Theory

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Hydro Simulations

- **Tough problem for hydro**
  - *Length scales vary drastically in time*
  - *Multiple fluids*
  - *Strongly time dependent viscosity*
  - *Very large number of time steps*
- **Special relativity, causality, …**
- **Huge magnetic fields**
- **3D simulations needed**
  - *Giant grids*

Simulations of Nuclear Collisions

- **Hydro, mean field, cascades**
- **Numerical solution of transport theories**
  - Need to work in 6d phase space => prohibitively large grids (20^3x40^2x80~10^9 lattice sites)
  - Idea: Only follow initially occupied phase space cells in time and represent them by test particles
  - One-body mean-field potentials (ρ, *p*, τ) via local averaging procedures
  - Test particles scatter with realistic cross sections => (exact) solution of Boltzmann equation (+Pauli, Bose)
  - Very small cross sections via perturbative approach
  - Coupled equations for many species no problem
  - Typically 100-1000 test particles/nucleon

Bertsch et al., Phys. Rev. C, '84

Equation to Solve

$$
\begin{aligned}
\frac{\partial f_b(xp)}{\partial t} + \frac{\Pi^i}{E_b^*(p)} \nabla_i^x f_b(xp) \quad &- \quad \frac{\Pi^\mu}{E_b^*(p)} \nabla_i^x U_\mu(x) \nabla_p^i f_b(xp) + \frac{M_b^*}{E_b^*(p)} \nabla_i^x U_s \nabla_p^i f_b(xp) \\
&= \quad I_{bb}^b(xp)
\end{aligned}
$$

$$
\begin{aligned}
I_{bb}^b(xp) \quad = \quad & \frac{\pi}{(2\pi)^9} \sum_{\alpha_1 \alpha_2 \alpha_3, m_s^b} \int\int\int d\boldsymbol{p}_1 d\boldsymbol{p}_2 d\boldsymbol{p}_3 \frac{M_b^* M_{\alpha_1}^* M_{\alpha_2}^* M_{\alpha_3}^*}{E_b^* E_{\alpha_1}^* E_{\alpha_2}^* E_{\alpha_3}^*} \\
\cdot \quad & \delta(E_b^*(p) + E_{\alpha_1}^*(p_1) - E_{\alpha_2}^*(p_2) - E_{\alpha_3}^*(p_3)) \delta(\boldsymbol{p} + \boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{p}_3) \\
\cdot \quad & \langle\langle p\alpha_b p_1 \alpha_1 | \hat{G} | p_2 \alpha_2 p_3 \alpha_3 \rangle\rangle \\
\cdot \quad & [\langle\langle p_2 \alpha_2 p_3 \alpha_3 | \hat{G} | p\alpha_b p_1 \alpha_1 \rangle\rangle - \langle\langle p_2 \alpha_2 p_3 \alpha_3 | \hat{G} | p_1 \alpha_1 p \alpha_b \rangle\rangle] \\
\cdot \quad & [f_{\alpha_2}(xp_2) f_{\alpha_3}(xp_3) \overline{f}_{\alpha_1}(xp_1) \overline{f}_b(xp) - \overline{f}_{\alpha_2}(xp_2) \overline{f}_{\alpha_3}(xp_3) f_{\alpha_1}(xp_1) f_b(xp)]
\end{aligned}
$$

Test Particles

- Baryon phase space function, f, is Wigner transform of density matrix
- Approximate formally by sum of delta functions, test particles

$$f(\vec{r},\vec{p},t) = \int d^3r_0\, d^3p_0\, \delta^3(\vec{r}-\vec{R}(\vec{r}_0,\vec{p}_0,t_0))\, \delta^3(\vec{p}-\vec{P}(\vec{r}_0,\vec{p}_0,t_0))\, f(\vec{r}_0,\vec{p}_0,t_0)$$

- Insert back into integral equation to obtain equations of motion for 6 coordinates of each test particle

Test Particle Equations of Motion

$$\frac{d}{dt}\vec{p}_i = -\vec{\nabla}U(\vec{r}_i) + \sum_{j\neq i} \frac{q_i\, q_j}{(\vec{r}_i - \vec{r}_j)^2} + \mathcal{C}(\vec{p}_i),$$

$$\frac{d}{dt}\vec{r}_i = \frac{\vec{p}_i}{\sqrt{m_i^2 + p_i^2}},$$

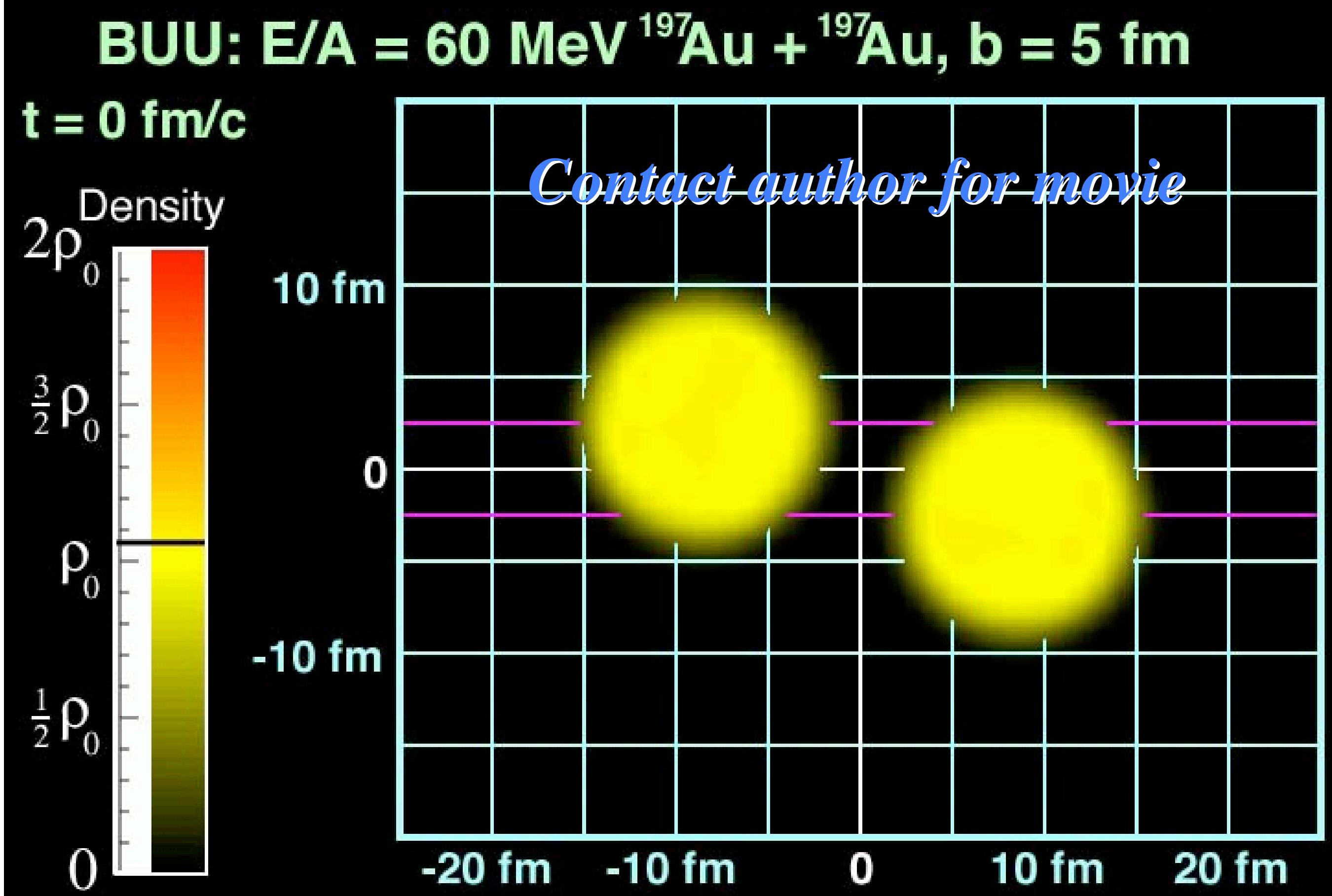
$$i = 1, \ldots, (A_t + A_p)\mathcal{N}$$

Coulomb

Nuclear EoS

Two-body scattering

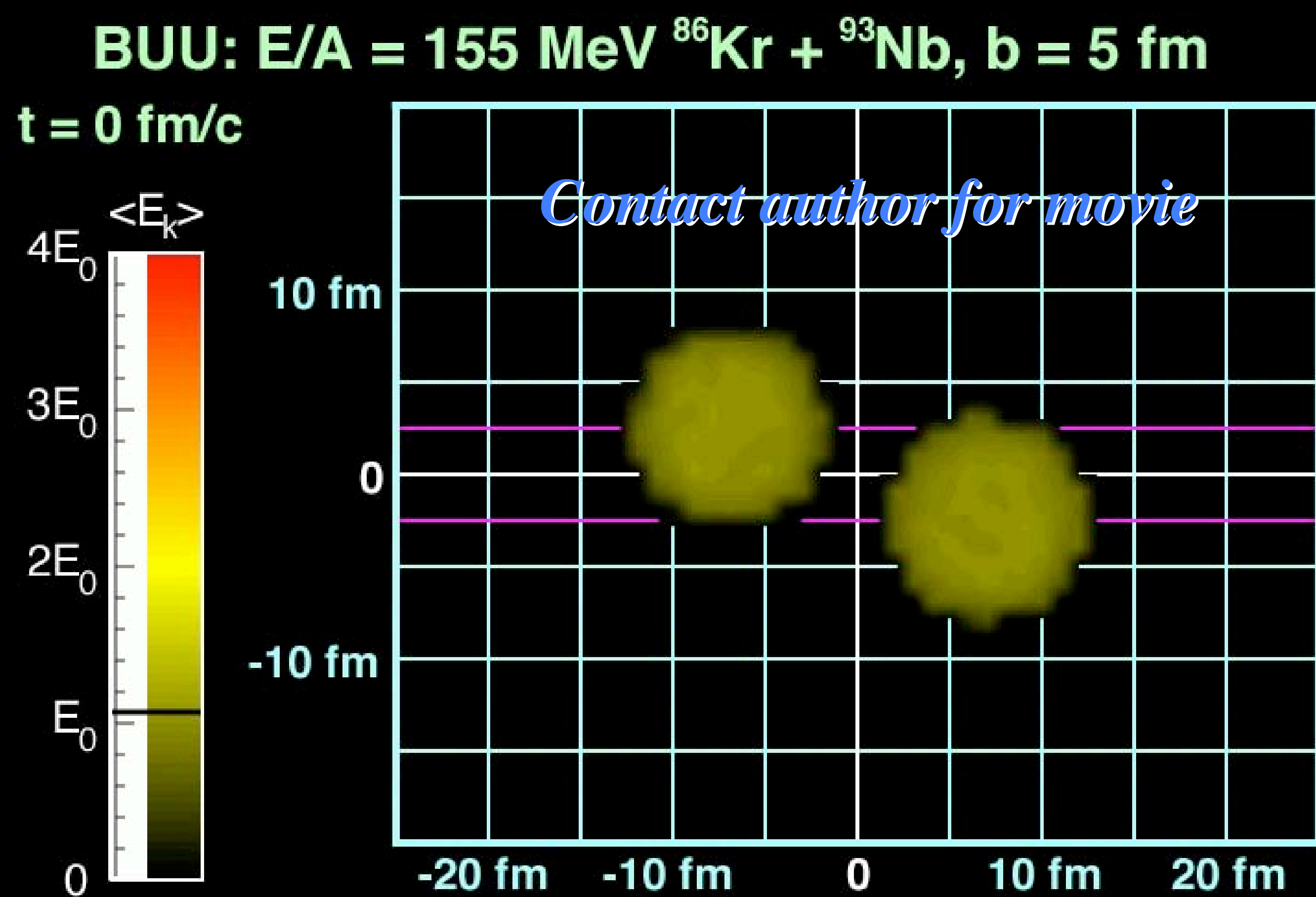
Example



- Density in reaction plane
- Integration over momentum space
- Cut for z=0+-0.5 fm

Momentum Space

- Output quantities (not input!)
- Momentum space information on
  - Thermalization & equilibration
  - Flow
  - Particle production
- Shown here: local temperature



Try this for Supernovae!

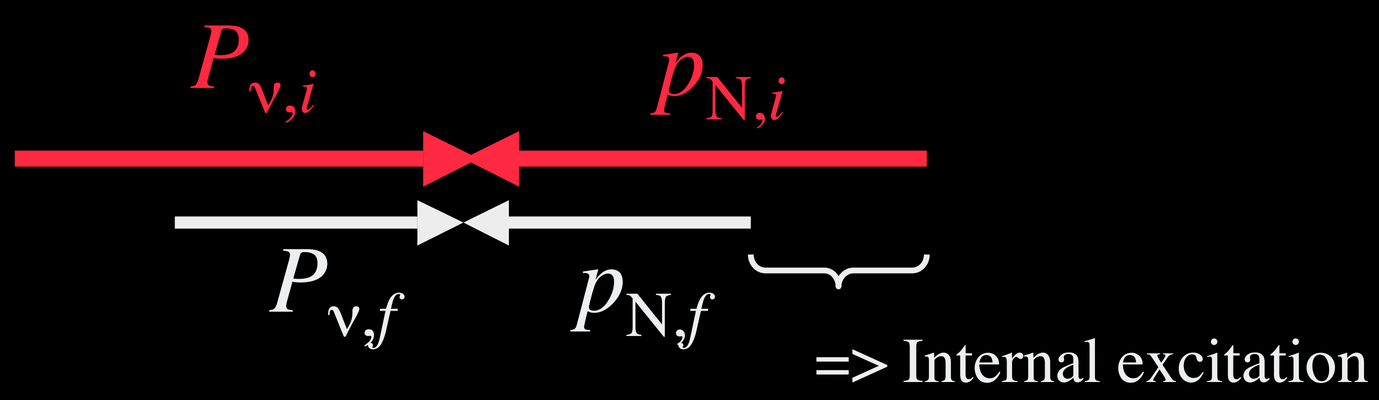
- 2 $M_\odot$ in iron core = 2×10^{57} baryons
- 10^7 test particles => 2×10^{50} baryons/test particle ☺
- Need time-varying grid for (non-gravity) potentials, because whole system collapses
- Need to think about internal excitation of test particles
- Can address angular momentum question [today]
- Can create ν-test particles and give them finite mean free path => Boltzmann solution for ν-transport problem [soon]
- Radiation transport via photon test particles (Bosons!)
- Should be able to treat magnetic fields self-consistently [next year]

Neutrinos

- **Neutrinos similar to pions at RHIC**
  - Not present in entrance channel
  - Produced in very large numbers (RHIC: 10^3, here 10^{56})
  - Essential for reaction dynamics
- **Different: No formation time or off -shell effects**
- **Represent 10^N neutrinos by one test particle**
  - Populate initial neutrino phase space uniformly
  - Sample test particle momenta from a thermal dist.
- **Neutrino test particles represent "3rd fluid", do NOT escape freely (neutrino trapping), and need to be followed in time.**
- **Neutrinos created in center and are "light" fluid on which "heavy" baryon fluid rests**
  - Inversion problem
  - Rayleigh-Taylor instability
  - turbulence

Neutrino Test particles

- Move on straight lines (no mean field)
- Scattering with hadrons
  - NOT negligible!
  - Convolution over all $\sigma_{A\nu} \propto A^2$ (weak neutral current)
  - Resulting test particle cross section angular distrib.: $\sigma_{cm}(\theta_f) = \delta(\theta_f - \theta_i)$
  - Center of mass picture:



Coupled Equations

$$\frac{\partial f_b(xp)}{\partial t} + \frac{\Pi^i}{E_b^*(p)}\nabla_i^x f_b(xp) - \frac{\Pi^\mu}{E_b^*(p)}\nabla_i^x U_\mu(x)\nabla_p^i f_b(xp) + \frac{M_b^*}{E_b^*(p)}\nabla_i^x U_s \nabla_p^i f_b(xp) = I_{bb}^b(xp) + I_{b\nu}^b(xp)$$

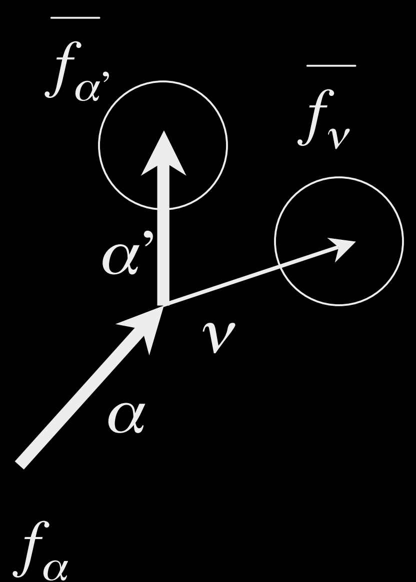
$$\frac{\partial f_\nu(xk)}{\partial t} + \frac{k\cdot\nabla^x}{E_\nu(k)} f_\nu(xk) = I_{b\nu}^\nu(xk)$$

$$\begin{aligned} I_{bb}^b(xp) = & \frac{\pi}{(2\pi)^9}\sum_{\alpha_1\alpha_2\alpha_3,m_s^b}\int\int\int d\boldsymbol{p}_1 d\boldsymbol{p}_2 d\boldsymbol{p}_3 \frac{M_b^* M_{\alpha_1}^* M_{\alpha_2}^* M_{\alpha_3}^*}{E_b^* E_{\alpha_1}^* E_{\alpha_2}^* E_{\alpha_3}^*} \\ & \cdot\ \delta(E_b^*(p) + E_{\alpha_1}^*(p_1) - E_{\alpha_2}^*(p_2) - E_{\alpha_3}^*(p_3))\delta(\boldsymbol{p} + \boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{p}_3) \\ & \cdot\ \langle\langle p\alpha_b p_1\alpha_1|\hat{G}|p_2\alpha_2 p_3\alpha_3\rangle\rangle \\ & \cdot\ [\langle\langle p_2\alpha_2 p_3\alpha_3|\hat{G}|p\alpha_b p_1\alpha_1\rangle\rangle - \langle\langle p_2\alpha_2 p_3\alpha_3|\hat{G}|p_1\alpha_1 p\alpha_b\rangle\rangle] \\ & \cdot\ [f_{\alpha_2}(xp_2) f_{\alpha_3}(xp_3)\overline{f}_{\alpha_1}(xp_1)\overline{f}_b(xp) - \overline{f}_{\alpha_2}(xp_2)\overline{f}_{\alpha_3}(xp_3) f_{\alpha_1}(xp_1) f_b(xp)] \end{aligned}$$

Similar to ***Wang, Li, Bauer, Randrup, Ann. Phys. '91***

Neutrino Gain and Loss

$$I^{\nu}_{b\nu}(\boldsymbol{r},\boldsymbol{k},t) = I^{\nu}_{\text{gain}}(xk) - I^{\nu}_{\text{loss}}(xk)$$



$$\bar{f} = 1 \pm f$$

$$\begin{aligned} I^{\nu}_{\text{gain}}(xk) &= \frac{\pi}{16(2\pi)^6} \sum_{\alpha\alpha'} \int\int \frac{M^*_{\alpha} M^*_{\alpha'}}{E^*_{\alpha}(p) E^*_{\alpha'}(p')} \\ &\cdot \frac{\langle u_{\alpha' p'} | \hat{\underline{u}}(k) \hat{u}(p+p')^2 | u_{\alpha p} \rangle \cdot \langle u_{\alpha p} | \hat{\underline{u}}(k) | u_{\alpha' p'} \rangle}{E^4_{\nu}(k)} \\ &\cdot \delta(E^*_{\alpha'}(p') - E_{\nu}(k) - E_{\alpha}(p))\ \delta(\mathbf{p}' - \mathbf{p} - \boldsymbol{k}) \\ &\cdot \bar{f}_{\nu}(xk) f_{\alpha'}(xp') \bar{f}_{\alpha}(xp)\ d\mathbf{p} d\mathbf{p}' \end{aligned}$$

$$\begin{aligned} I^{\nu}_{\text{loss}}(xk) &= \frac{\pi}{16(2\pi)^6} \sum_{\alpha\alpha'} \int\int \frac{M^*_{\alpha} M^*_{\alpha'}}{E^*_{\alpha}(p) E^*_{\alpha'}(p')} \\ &\cdot \frac{\langle u_{\alpha' p'} | \hat{\underline{u}}(k) \hat{u}(p+p')^2 | u_{\alpha p} \rangle \cdot \langle u_{\alpha p} | \hat{\underline{u}}(k) | u_{\alpha' p'} \rangle}{E^4_{\nu}(k)} \\ &\cdot \delta(E^*_{\alpha'}(p') - E_{\nu}(k) - E_{\alpha}(p))\ \delta(\mathbf{p}' - \mathbf{p} - \boldsymbol{k}) \\ &\cdot f_{\nu}(xk) f_{\alpha}(xp) \bar{f}_{\alpha'}(xp')\ d\mathbf{p} d\mathbf{p}' \end{aligned}$$

Numerical Realization

- Test particle equations of motion

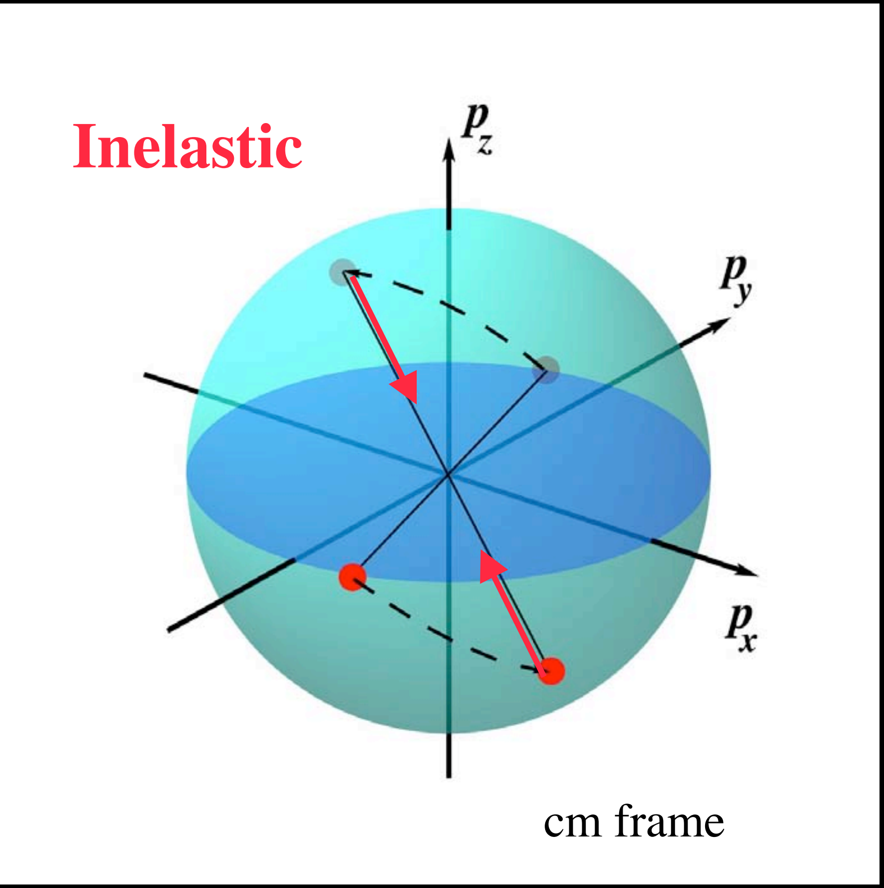
$$\begin{aligned}\frac{d}{dt}\vec{p}_j &= -\vec{\nabla}U_{\mathrm{EoS},e^-}(\vec{r}_j) + \vec{F}_{G,j}(\vec{r}_1,\ldots,\vec{r}_{N_{tp}}) + \mathcal{C}(\vec{p}_j) + \mathcal{C}_\nu(\vec{p}_j)\\ \frac{d}{dt}\vec{r}_j &= \frac{\vec{p}_j}{\sqrt{m_{tp}^2 + p_j^2}}\\ j &= 1,\ldots,N_{tp}\end{aligned}$$

- Nuclear EoS evaluated on spherical grid
- Newtonian monopole approximation for gravity

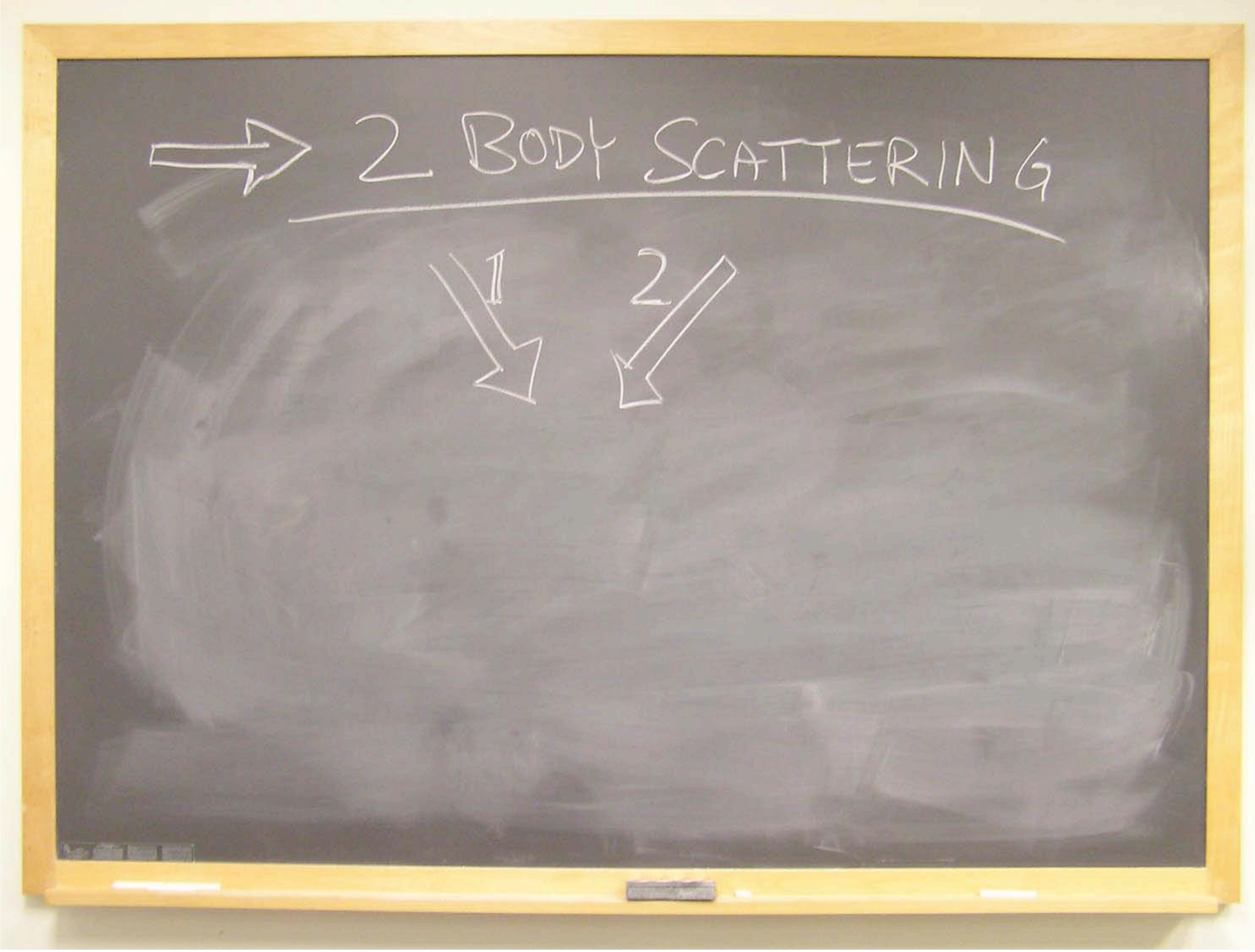
$$\vec{F}_{G,j} = -G\,\frac{m_{tp}^2\,\#\left\{i\in\{1,\ldots,N_{tp}\} : |\vec{r}_i| < |\vec{r}_j|\right\}}{|\vec{r}_j|^3}\vec{r}_j.$$

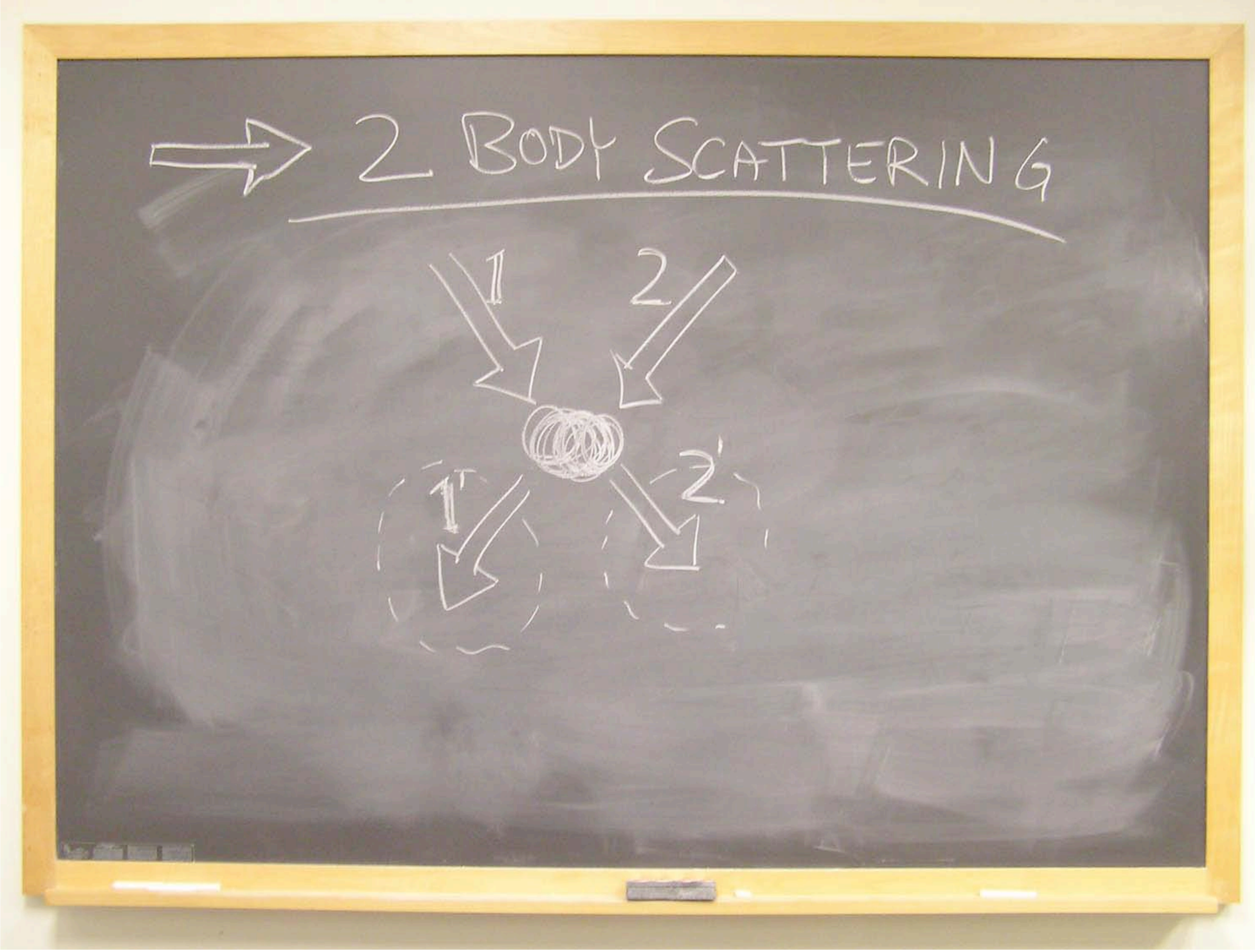
Test Particle Scattering

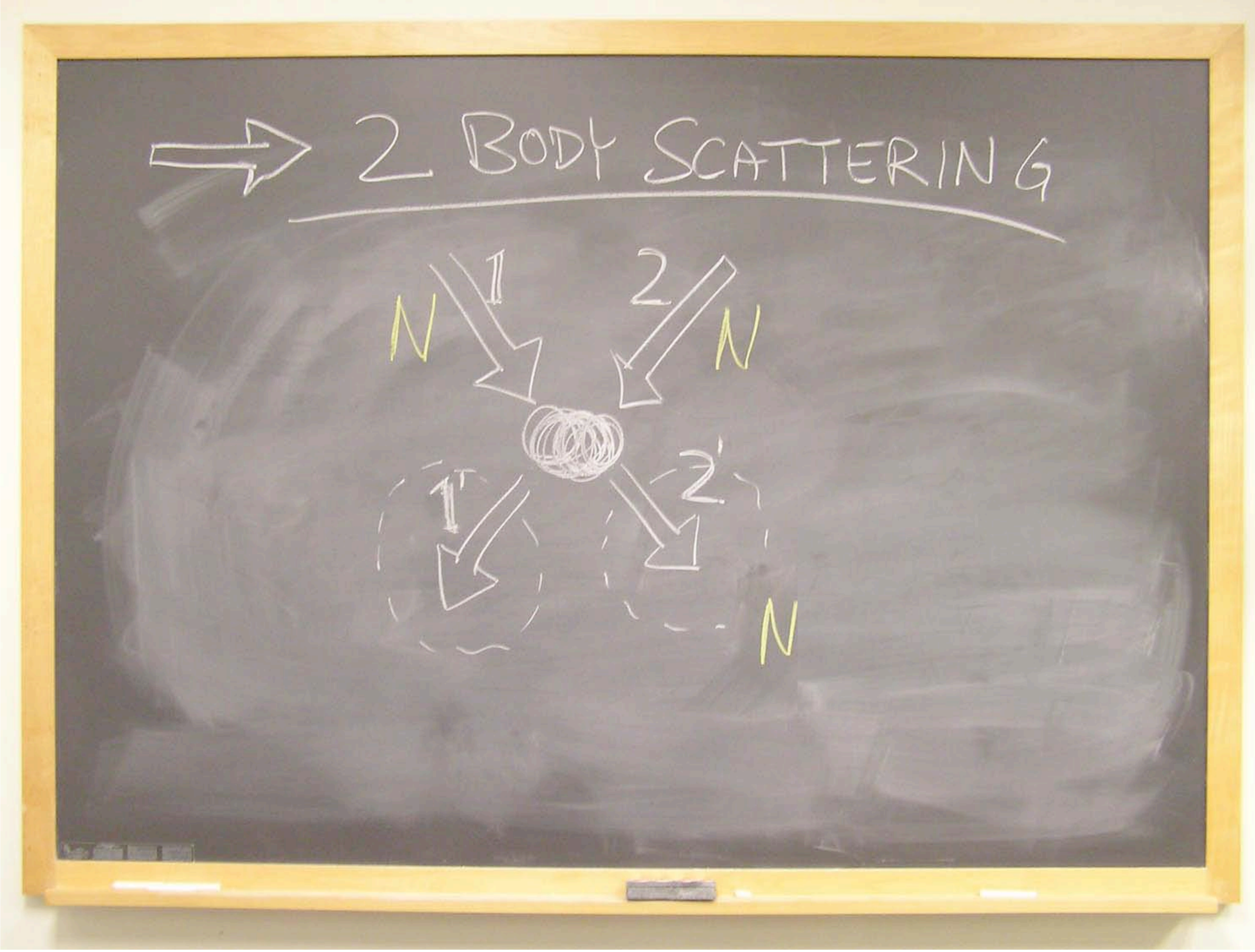
- Nuclear case: test particles scatter with (reduced, due to Pauli Principle) nucleon-nucleon cross sections
  - Elastic and inelastic



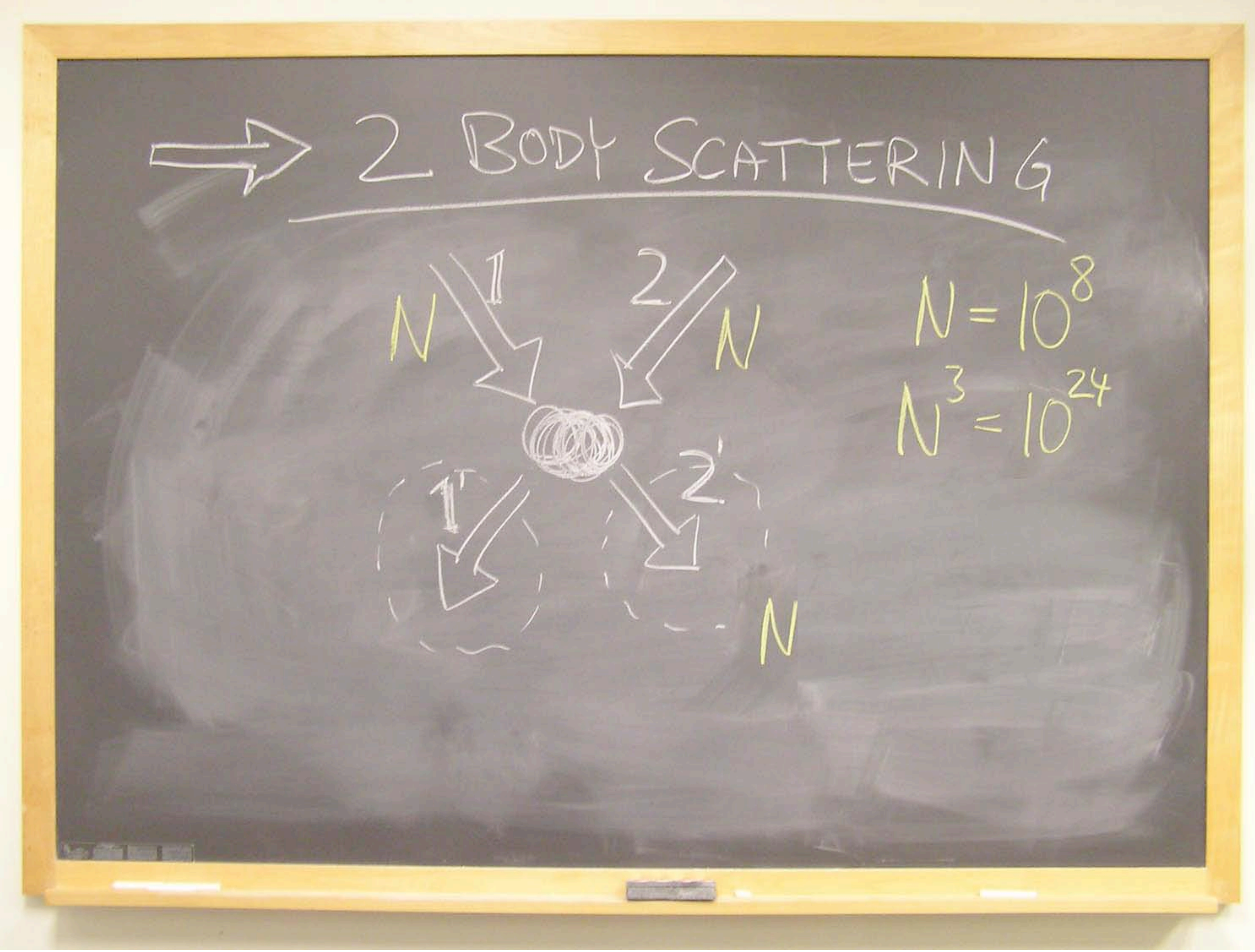
- Similar rules apply for astro test particles
  - Scale invariance
  - Shock formation
  - Internal heating

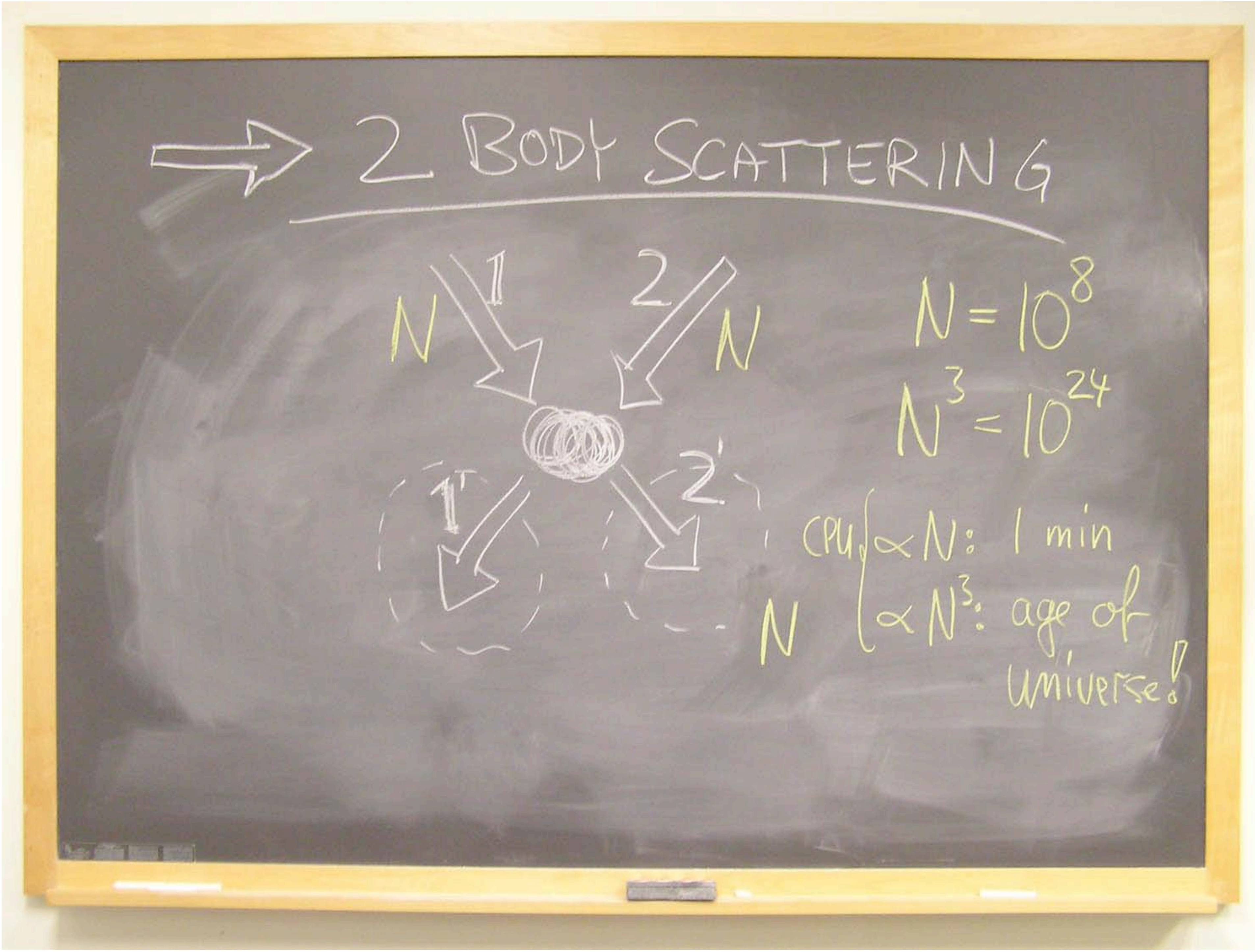

2 BODY SCATTERING
1
2


2 BODY SCATTERING
1
2
1'
2'



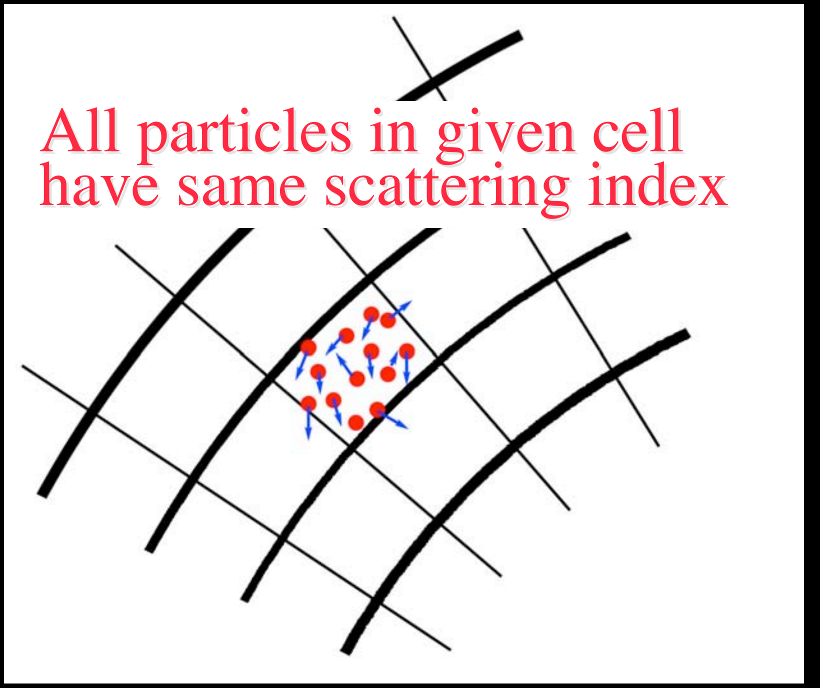
2 BODY SCATTERING
1
2
N
N
1'
2'
N


2 BODY SCATTERING
1
2
N
N
N = 10^8
N^3 = 10^24
1'
2'
N


2 BODY SCATTERING
1
2
N
N
1'
2'
N = 10^8
N^3 = 10^24
CPU
∝ N: 1 min
∝ N^3: age of universe!
N

DSMC

- **Stochastic Direct Simulation Monte Carlo**
  - Do not use closest approach method
  - Randomly pick ***k*** collision partners from given cell
  - Redistribute momenta within cell with fixed i_r, i_θ, i_ϕ
- **Technical details:**
  - QuickSort on scattering index of each particle makes CPU time consumption ~ $k\,N \log N$
  - Final state phase space approximated by local *T* Fermi-Dirac (no additional power of *N*)
  - Hydro limit: just generate "enough" collisions, no need to evaluate matrix elements



Excluded Volume

- Collision term simulation via stochastic scattering (Direct Simulation Monte Carlo)
  - Additional advection contribution (EoS modification)

$$\vec{d} = \frac{1}{2\sqrt{\pi}}\sqrt{\sigma_{NN}(E)} \cdot \vec{e}_{v-v'}$$

  - Modification to collision probability

$$\frac{P'}{P} = \frac{1 - \frac{11}{8} b^E \rho(r)}{1 - 2 b^E \rho(r)}; \quad b^E = \frac{2}{3}\pi a^3$$

Shadowing

Excluded V

= 2nd Enskog virial coefficient

$$a = \sqrt{\frac{1}{4\pi n}\sum_{i=1}^{n} \sigma_i \cdot \left(1 - 2\frac{p_F^3}{(p_F + p_B)^3}\right)}$$

Pauli

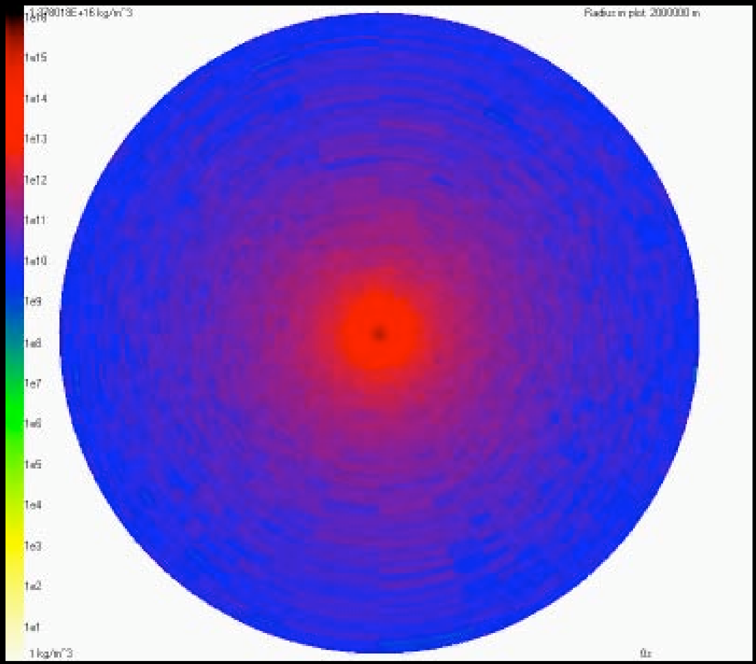
free

= hard sphere radius

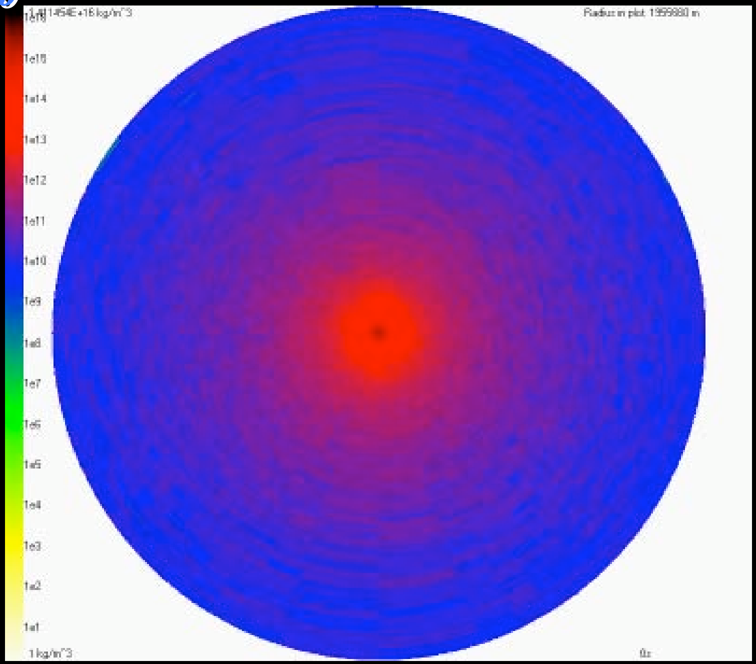
Alexander, Garcia, Alder, PRL '95
Kortemeyer, Daffin, Bauer, PRB '96

Effect of Baryon-Baryon Collisions on Dynamics

Contact author for movies



Mean field only



Hydro limit

Parameters

$N = 10^6$

Grid: $90_r * 40_\phi * 40_\theta$

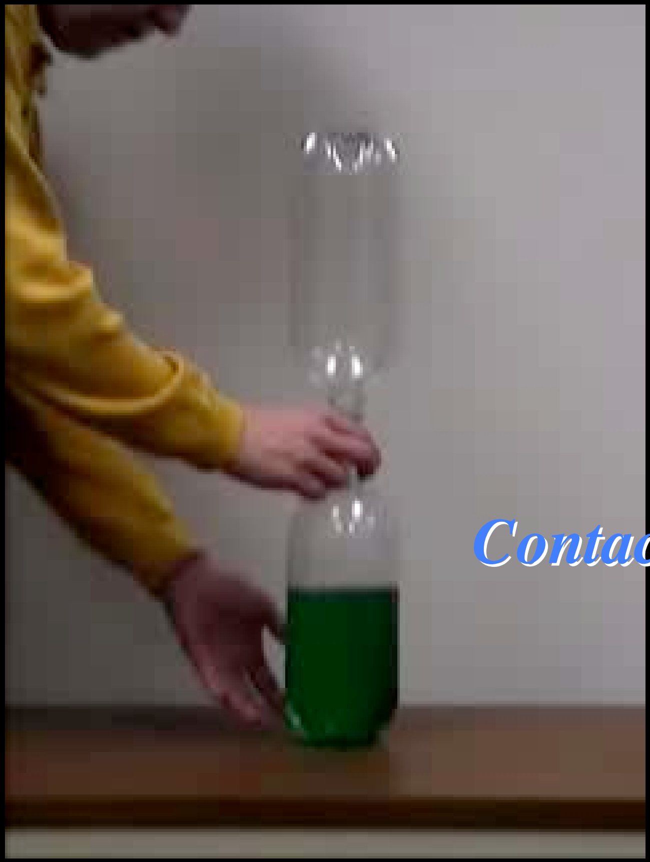
$R_\mathrm{i} = 2$ km

$M = 2\,M_\odot$

First Rotating Results

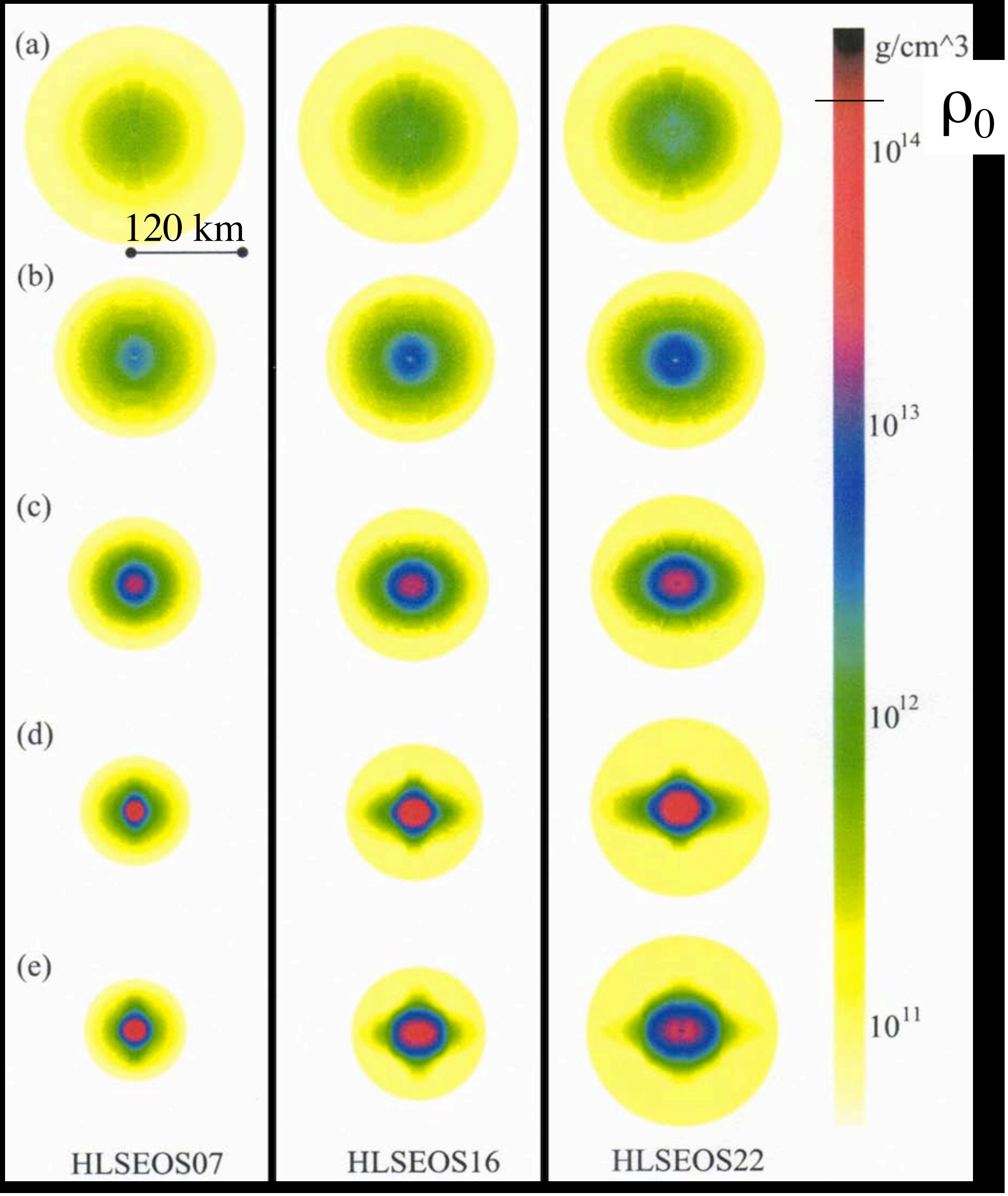
- **Mean field level**
  - Only nuclear and electron gas EoS, gravity
  - No collisions yet
- **Exploratory: role of collective rotation**

Effects of Angular Momentum



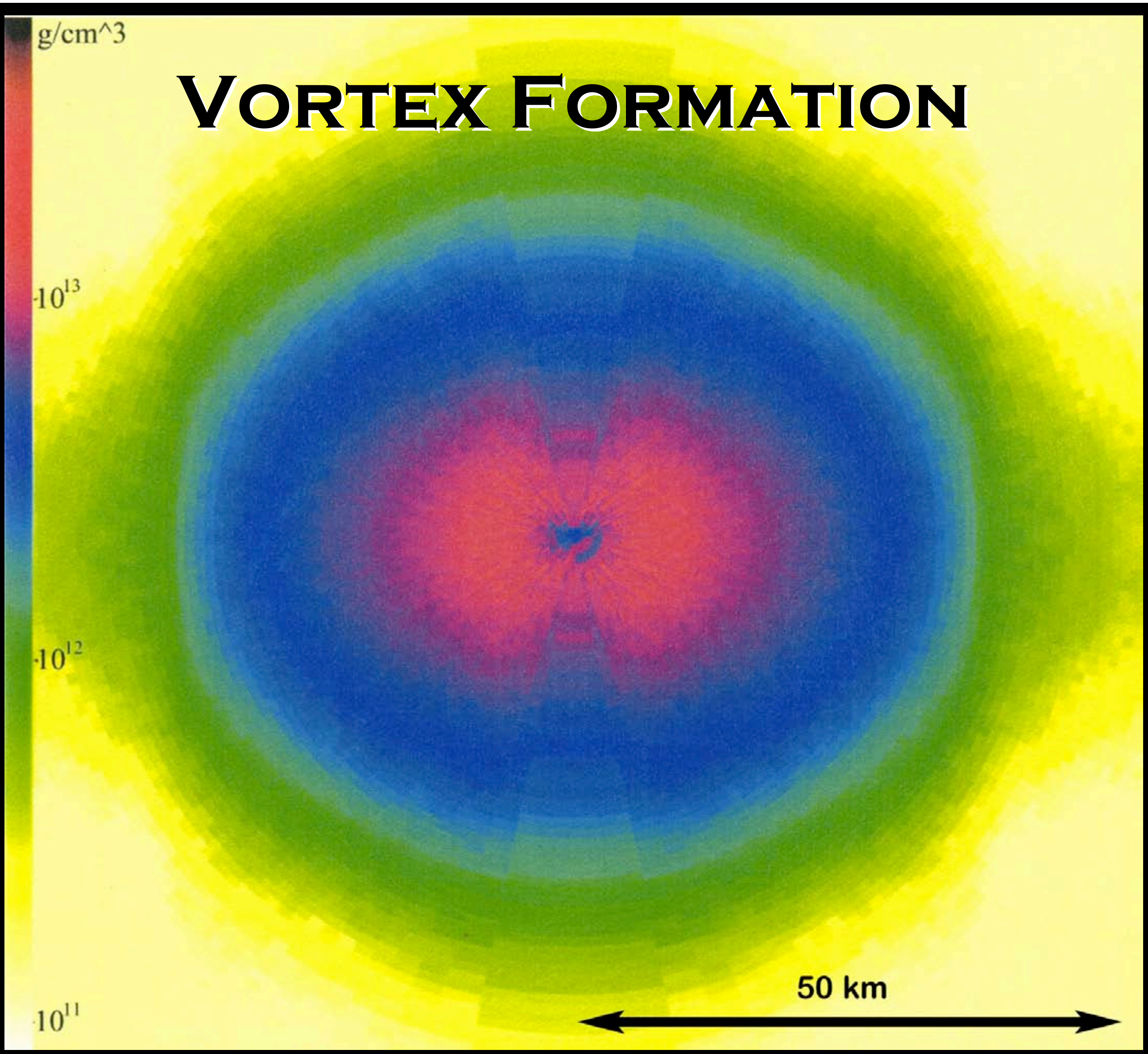
Contact author for movies

(a) Initial conditions
(b) After 2 ms
(c) After 3 ms
(d) Core bounce
(e) 1 ms after core bounce



W. Bauer, INT 2004

Vortex Formation



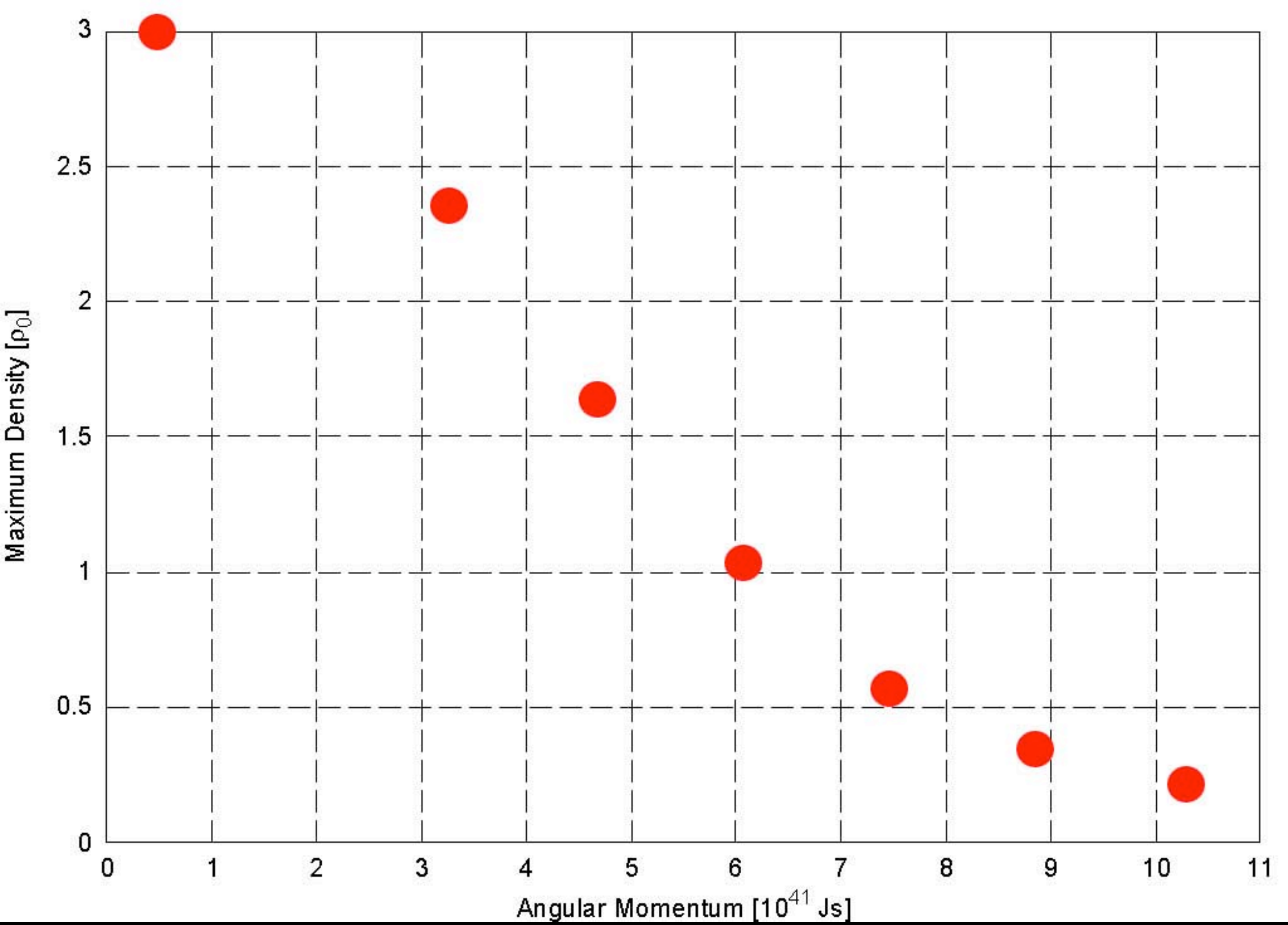
Results

- "mean field" level
- 1 fluid: hadrons

Name of Run	ω_0 $[\frac{\text{rad}}{\text{s}}]$	$\lvert\vec{L}_0\rvert$ $[10^{41}\,\text{Js}]$	$\lvert\frac{E_{rot}}{E_G}\rvert_{init}$	t_{bounce} [ms]	ρ_{max} $[\rho_0]$
HLSEOS01	10	0.466	0.027%	3.44	2.99
HLSEOS07	70	3.26	1.3%	3.61	2.35
HLSEOS10	100	4.67	2.7%	3.76	1.63
HLSEOS13	130	6.06	4.5%	3.89	1.03
HLSEOS16	160	7.46	6.8%	4.03	0.56
HLSEOS19	190	8.86	9.6%	4.17	0.34
HLSEOS22	220	10.3	13%	4.31	0.21

Max. Density vs. Angular Momentum

- Mean field only!!!



The People who Actually do the Work

Tobias Bollenbach

(M.S. Thesis, MSU, 2002, now in Dresden, Germany)

Terrance Strother →

MSU graduate student

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