NOTRE DAME – OCT. '04

R - MATRIX THEORY

INTRODUCTION

1 Origins

CHADWICK, FERMI, BREIT-WIGNER, SHELL MODEL

2 Reaction Theory Frameworks

KAPUR & PEIERLS (1938)
WIGNER & EISENBUD (1941-47)
ROSENFELD & HUMBLET
FESHBACH

3 Why is the R-matrix Framework the best for resonance reactions?

Compound Nucleus with well defined radius
Parameters tied to nuclear spectroscopy

4. Approximations within the Framework

Many Levels (few channels)
Many Channels (few levels)

5 Basic Concepts

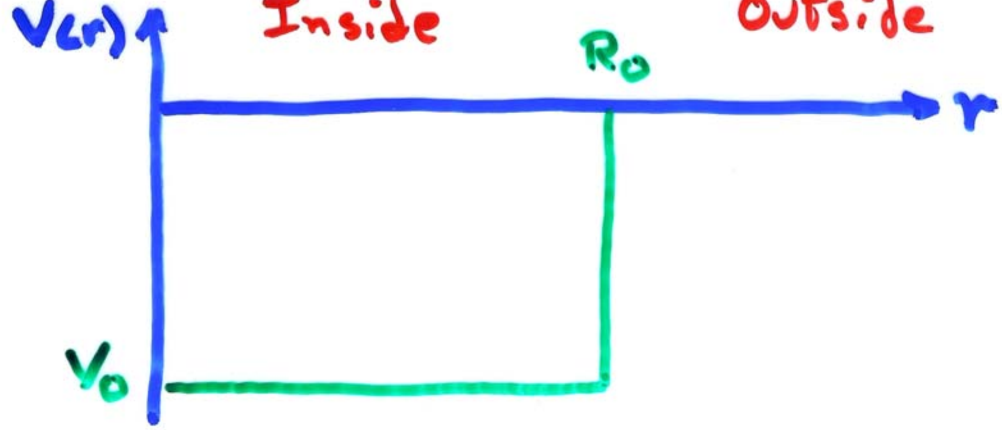
"External": Cross Sections, Phase Shifts
Collision Matrix

"Internal": Level Energies
Level Widths
Reduced widths

Auxiliary Parameters: channel radii
channel boundary conditions.

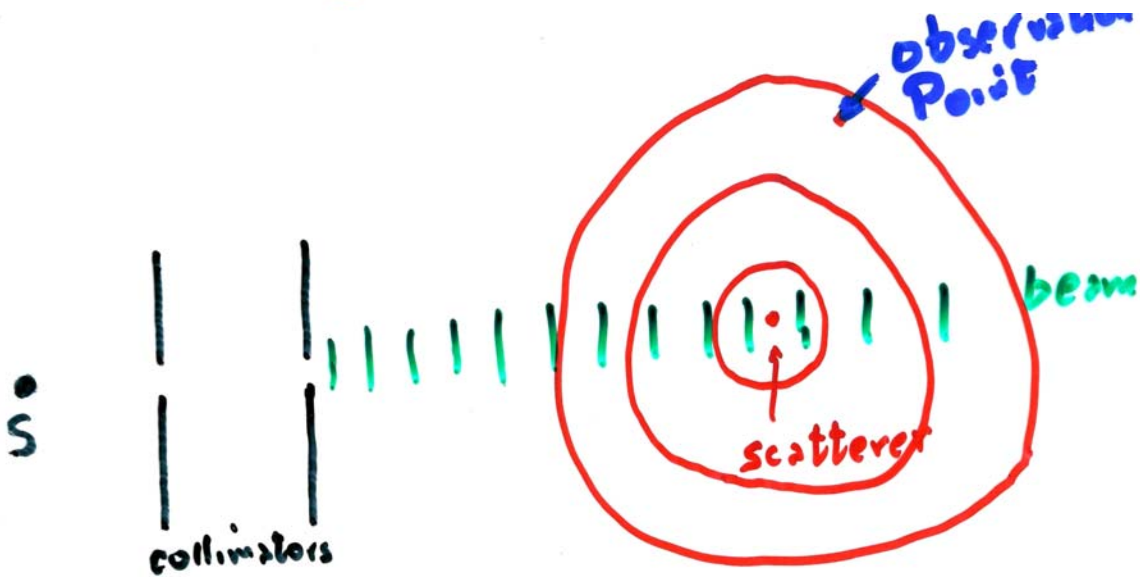
Example

s-wave neutrons scattered by a Square Well



To be semi-realistic

$$V_0 \approx -50\,\text{MeV}$$
$$R_0 \approx 1.4A^{1/3}\ \text{fm.}$$



EXTERNAL BITS (schiff)

$$\psi(r,\theta,\phi) = A\left\{ \underset{\Uparrow}{e^{ikz}} + \frac{1}{r} \underset{\Uparrow}{f(\theta,\phi)e^{ikr}} \right\}$$

$k = \frac{mv}{\hbar}$ incident plane wave scattered wave

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} (2\ell+1)\, i^{\ell}\, j_\ell(kr)\, P_\ell(\cos\theta)$$

$$\sigma(\theta,\phi) = |f(\theta,\phi)|^2$$

for $\ell = 0$, $j_0(kr) = \frac{1}{kr}\sin kr = \frac{2i}{kr}\left(\underset{\substack{\Downarrow \\ I_0}}{e^{-ikr}} - \underset{\substack{\Downarrow \\ O_0}}{e^{ikr}}\right)$

If there is scattering the outgoing wave is phase shifted

radial solution $\underset{r\to\infty}{:}$ $A_\ell \sin(kr + \delta_\ell)$

$$\to 2i A_\ell e^{-i\delta_\ell}\left[I_\ell - e^{2i\delta_\ell} O_\ell\right]$$

$$U_\ell \equiv e^{2i\delta_\ell} \qquad \left[I_\ell - U_\ell O_\ell\right]$$

$$\sigma = \frac{\pi}{k^2}|1 - U_\ell|^2$$

For a spherically-symmetric potential

$$f(\theta,\phi) \equiv f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\left(e^{2i\delta_l}-1\right)P_l(\cos\theta)$$

$$\frac{d\sigma}{d\theta} \equiv |f(\theta)|^2 = \frac{1}{k^2}\left|\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l\, P_l(\cos\theta)\right|^2$$

$$\sigma \equiv \int_0^{\pi}\left(\frac{d\sigma}{d\theta}\right)\sin\theta\, d\theta = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

INTERNAL BITS

radial equation $$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dr^2} + V\phi = E\phi \qquad (1)$$

$$V = -V_0 \quad r \le R_0,$$
$$= 0 \quad r \ge R_0$$

1st Boundary Condition : ϕ must be "regular"

INTERNAL $$\phi(r) = A\sin Kr \quad (2) \qquad E_0 + V_0 = \frac{\hbar^2 K^2}{2m}$$

EXTERNAL $$\phi(r) = \underbrace{I(r)}_{\frac{1}{\sqrt{4\pi v}}e^{-ikr}} - e^{2i\delta}\underbrace{O(r)}_{\frac{1}{\sqrt{4\pi v}}e^{ikr}} \qquad (3)$$

Matching: at $r = R_0$ $$\phi'/\phi\,\Big|_{internal} = \phi'/\phi\,\Big|_{external}$$

$$\frac{K\cos KR_0}{\sin KR_0} = \frac{ik\left[-e^{-ikR_0} - e^{2i\delta}e^{ikR_0}\right]}{\left[e^{-ikR_0} - e^{2i\delta}e^{ikR_0}\right]}$$

$$\Downarrow$$

$$\delta = \tan^{-1}\left[\frac{k}{K}\tan KR_0\right] - kR_0$$

$$\sigma = (4\pi/k^2)\sin^2\delta$$

σ has a maximum whenever $\delta = (n+\frac{1}{2})\pi$

Is this a "resonance" ??

The Square Well as a Resonant Cavity

To display the resonances we need the radial wave equation & a second boundary condition.

$$-\frac{\hbar^2}{2m}\frac{d^2 X_\lambda}{dr^2} + V X_\lambda = E_\lambda X_\lambda \qquad r \le R_0 \qquad (4)$$

2nd boundary condition

$$r\frac{dX_\lambda}{dr}\Big|_{r=R_0} \equiv b\, X_\lambda(R_0) \qquad b \text{ is real.} \qquad (5)$$

then

$$X_\lambda(r) = \sqrt{\frac{2}{R_0}}\,\sin(K_\lambda r) \qquad \text{if } b=0 \qquad (6)$$

$$K_\lambda = (\lambda + \tfrac{1}{2})\,\pi/R_0 \qquad (7)$$

Fourier decomposition of any $\phi(r)$

$$\phi(r) = \sum_\lambda C_\lambda X_\lambda(r) \qquad (8)$$

$$C_\lambda = \int_0^{R_0} X_\lambda^*(r)\,\phi(r)\,dr \qquad (9)$$

Multiply (1) by X_λ^* and the c.c. of (4) by $\phi(r)$, subtract and integrate to obtain:

$$\left(\frac{\hbar^2}{2m}\right)\left[\phi\frac{dX_\lambda^*}{dr} - X_\lambda^*\frac{d\phi}{dr}\right]_{r=R_0} = (E - E_\lambda)\int_0^{R_0}\phi\, X_\lambda^*\, dr \qquad (10)$$

yielding:

$$C_\lambda = \frac{1}{(E_\lambda - E)}\left(\frac{\hbar^2}{2mR_0}\right) X_\lambda^*(R_0)\left[\phi'(r_0) - b\,\phi(R_0)\right] \qquad (11)$$

$$\phi' \equiv \frac{d}{dr}\phi$$

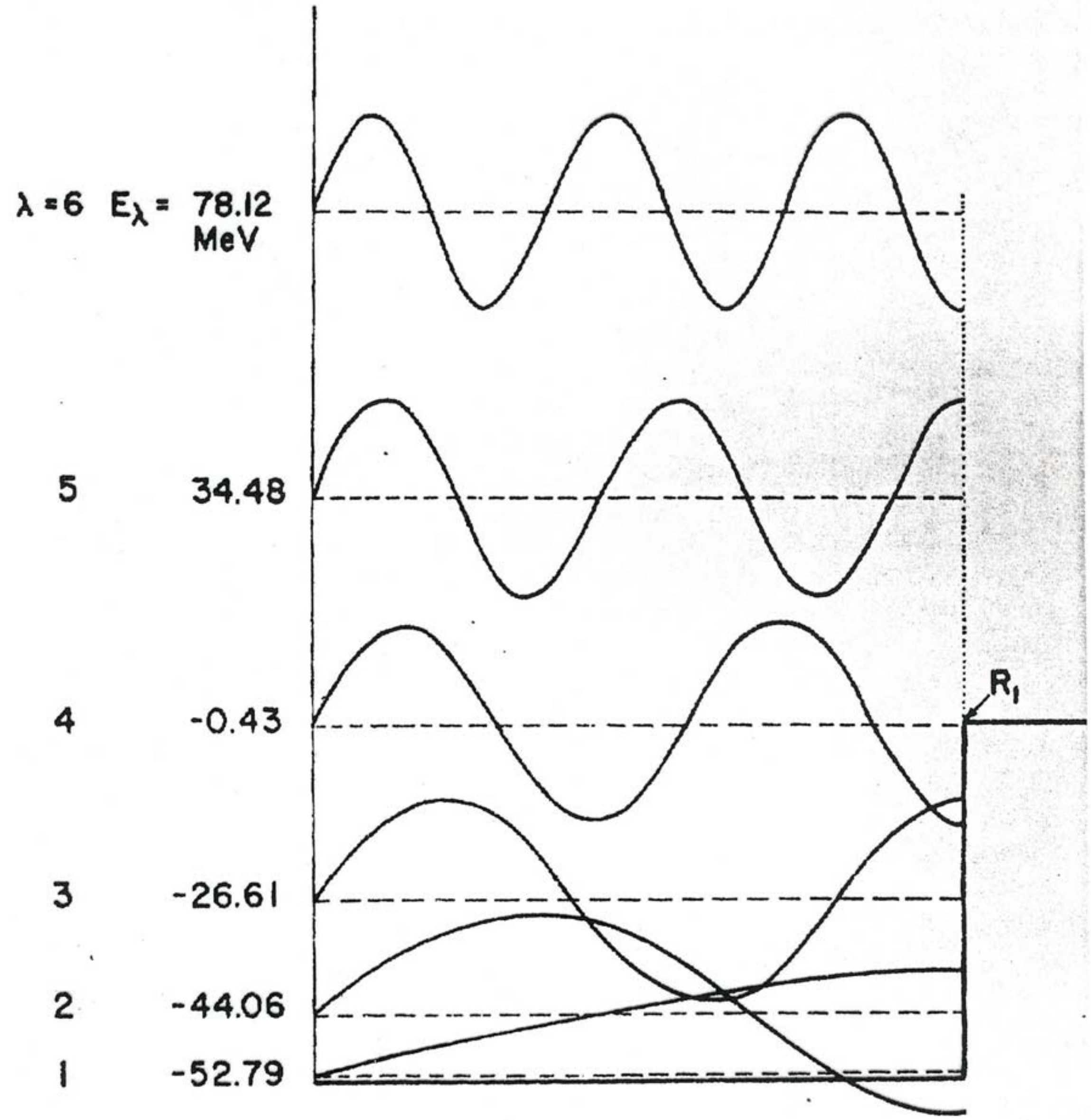


FIG. 2. The first six standing waves of the square we
structed with a boundary condition number ($b = 0$) ap

$\mathcal{R}$-function & Collision Function, $\mathcal{U}$

Inserting (11) into (8) yields:

$$\phi(R_0) = \mathcal{R}\left[\phi'(R_0) - b\,\phi(R_0)\right] \qquad (12)$$

where

$$\boxed{\mathcal{R} \equiv \sum_\lambda \frac{\gamma_\lambda^2}{E_\lambda - E}} \qquad (13)$$

with

$$\gamma_\lambda^2 \equiv \frac{\hbar^2}{2mR_0}\left|X_\lambda(R_0)\right|^2 \qquad (14)$$

$$= \frac{\hbar^2}{mR_0^2} \quad \text{if } b=0$$

$\leftarrow$ single particle reduced width

If we look at

$$\delta = \tan^{-1}\left[kR_0 \frac{1}{KR_0}\tan KR_0\right] - kR_0$$

"internal bits" $\rightarrow$

the $\mathcal{R}$-function is the Fourier expansion of the "internal bits"

$$\delta = \tan^{-1}\left[kR_0 \frac{\mathcal{R}}{1+b\mathcal{R}}\right] - kR_0$$

Connecting "Internal" & "External" bits

Internally: $\dfrac{\phi'(R_0)}{\phi(R_0)} = \dfrac{1+bR}{R}$.

Externally: $\dfrac{\phi'(R_0)}{\phi(R_0)} = \left.\dfrac{I'-UO'}{I-UO}\right|_{r=R_0}$

Equating these yields:

$$\boxed{U = O^{-1}[1-RL]^{-1}[1-RL^*]\,I}$$

with $L \equiv O'O^{-1} - b$

$= -b + ikR_0$

(all O, O', I, I', L evaluated at $r = R_0$)

for our case of s-wave neutrons

$$U = e^{-2ikR_0}\,\frac{1+bR-ikR_0R}{1+bR-ikR_0R}$$

Breit-Wigner Formula

If we approximate $R \simeq \dfrac{\gamma_\lambda^2}{E_\lambda - E}$ only one resonance

then:

$$\sigma = \frac{\pi}{k^2}\left|2\sin kR_0\, e^{ikR_0} - \frac{\Gamma_\lambda}{(E_\lambda - E + \Delta_\lambda) - \frac{i}{2}\Gamma_\lambda}\right|^2$$

with $\Gamma_\lambda \equiv 2\underbrace{kR_0}_{\rho}\,\gamma_\lambda^2$, $\Delta \equiv b\gamma_\lambda^2$

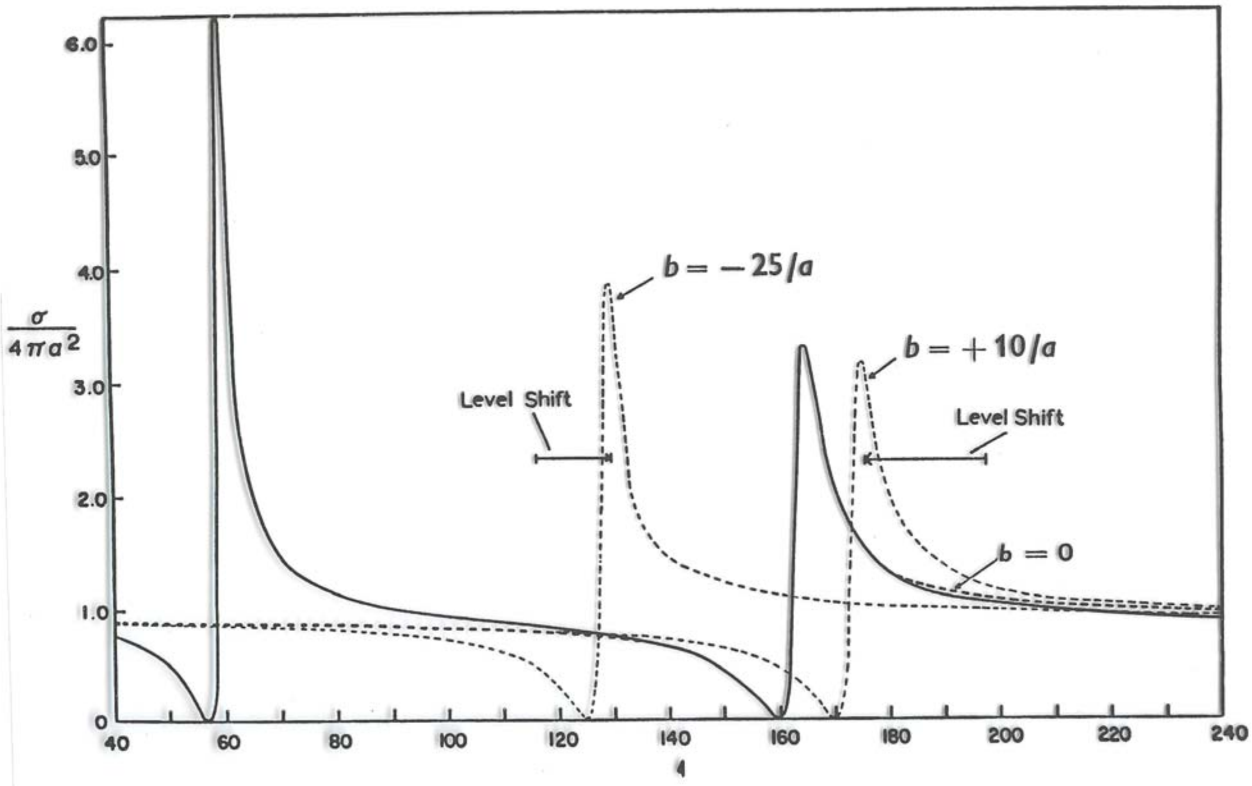


Fig. 1. The s-wave scattering cross sections for a nucleon by a square well of depth 40 MeV and a width equal to the nuclear radius $a(= 1.45A^{1/3} \times 10^{-13}$ cm, where A is the atomic weight). The solid line gives the exact cross section (divided by $4\pi a^2$) at 100 keV in terms of the atomic weight A. Thus the abscissa is related to the width of the well not to the energy of the particle. The broken lines give the one level approximation to the cross section for various values of the arbitrary boundary condition number b, of the resonance theory. The heavy broken line, corresponding to $b = 0$, merges with the exact curve near the resonance.