Formal and Physical $\boldsymbol{R}$-matrix parameters

Carl R. Brune
Edwards Accelerator Laboratory, Department of Physics and Astronomy, Ohio University, Athens OH 45701, USA
(Dated: Octer 21, 2004)

Notes from 11 October 2004 lecture presented at the Joint Institute for Nuclear Astrophysics $\boldsymbol{R}$-Matrix School at Notre Dame University.

I. INTRODUCTION

The $\boldsymbol{R}$-matrix theory of reactions has proven over the course of time to be very useful in nuclear and atomic physics, both for the fitting of experimental data and as a tool for theoretical calculations. In these notes I will explore the relationship between formal $\boldsymbol{R}$-matrix parameters and "physical" resonance parameters.

II. HOW DOES ONE DEFINE A RESONANCE ENERGY?

In practice one is likely to encounter many definitions of "*the* resonance energy" – e.g. the energy where there is a peak of an excitation function, the energy when the phase shift is equal to $\pi/2$ (for elastic scattering), the complex poles of the scattering matrix,... Many approaches involve the Breit-Wigner resonance formula:

$$\sigma_{cc'}(E) = \frac{\pi\omega}{k_c^2}\frac{\Gamma_c^o\Gamma_{c'}^o}{(E_R - E)^2 + (\Gamma^o/2)^2} \tag{1}$$

where $c(c')$ label the incoming (outgoing) channel, k_c^2 is the incoming wavenumber, $\omega = (2J+1)/[(2J_1+1)(2J_2+1)]$ is the statistical factor, E is the (variable) reaction energy, E_R is the resonance energy, Γ_c^o are the *observed* partial widths of the resonance, and the total *observed* width Γ^o is given by sum over all channels of the partial widths: $\Gamma^o = \sum_c \Gamma_c^o$. It should be noted that the Breit-Wigner formula is a very general quantum-mechanical result with applications ranging from atomic physics to particle physics (e.g. the 92-GeV resonance in $e^+ - e^-$ scattering known as the Z_0 boson). In some cases the partial widths may be energy dependent. From a time-dependent viewpoint the resonant state decays with time-dependence $\exp(-t/\tau)$, where τ is the mean lifetime and $\Gamma^o\tau = \hbar$. I will describe below one approach for defining resonance energies and partial widths in the $\boldsymbol{R}$-matrix formalism which is particularly convenient. The differences between various prescriptions for defining resonance energies and partial widths are generally most significant for broad resonances. Let me conclude this section with a caution: since different methods are in use for defining resonance parameters, it is very important to (1) understand what conventions are used by others if you use their results and (2) supply enough information so that others may properly use your results.

III. REVIEW OF $\boldsymbol{R}$-MATRIX THEORY

I will begin by reviewing some of the notation and results of $\boldsymbol{R}$-matrix theory as described by Lane and Thomas (LT) [1] which will be needed later. The $\boldsymbol{R}$ matrix is a function of the energy E and is defined by

$$R_{c'c} = \sum_\lambda \frac{\gamma_{\lambda c'}\gamma_{\lambda c}}{E_\lambda - E}, \tag{2}$$

where E_λ are the level energies, $\gamma_{\lambda c}$ are the reduced width amplitudes, λ is the level label, and c is the channel label. I will assume that the numbers of levels and channels are finite and given by N_λ and N_c, respectively. One must also specify the constant(s) B_c, which determine the boundary conditions satisfied by the underlying eigenfunctions. I will also consider here a set of levels which all have the same J^π value (the presence of additional levels with other J^π values would not impact any results discussed here).

When working in the level-matrix framework, it is convenient to define column vectors $\boldsymbol{\gamma}_c$ in level space from the $\gamma_{\lambda c}$:

$$\boldsymbol{\gamma}_c = \begin{pmatrix} \gamma_{1c} \\ \gamma_{2c} \\ \gamma_{3c} \\ \cdots \end{pmatrix} \tag{3}$$

as well as a diagonal matrix $\boldsymbol{e}$ containing the E_λ:

$$[\boldsymbol{e}]_{\lambda\mu} = E_\lambda \delta_{\lambda\mu}. \tag{4}$$

The $N_\lambda \times N_\lambda$ level matrix $\boldsymbol{A}$ can now be defined by its inverse:

$$[\boldsymbol{A}^{-1}]_{\lambda\mu} = (E_\lambda - E)\delta_{\lambda\mu} - \sum_c \gamma_{\lambda c}\gamma_{\mu c}(S_c + \mathrm{i}P_c - B_c), \text{ or} \tag{5}$$

$$\boldsymbol{A}^{-1} = \boldsymbol{e} - E\mathbf{1} - \sum_c \boldsymbol{\gamma}_c \boldsymbol{\gamma}_c^T (S_c + \mathrm{i}P_c - B_c), \tag{6}$$

where $\mathbf{1}$ is the unit matrix and P_c and S_c are the energy-dependent penetration and shift functions. Note also the notation $\boldsymbol{X}^T \equiv \text{transpose}(\boldsymbol{X})$ for any matrix $\boldsymbol{X}$. The collision matrix $\boldsymbol{U}$ which determines the cross sections can be calculated from the $\boldsymbol{A}$ matrix:

$$[\boldsymbol{U}]_{c'c} = \Omega_{c'}\Omega_c \left[\delta_{c'c} + 2\mathrm{i}(P_{c'}P_c)^{1/2}\boldsymbol{\gamma}_{c'}^T \boldsymbol{A}\boldsymbol{\gamma}_c\right], \tag{7}$$

where the Ω_c are defined in terms of Coulomb wavefunctions (see LT, Eq. III.4.5).

IV. THE SINGLE-LEVEL *R*-MATRIX

In order to interpret the meaning of the $\boldsymbol{R}$-matrix parameters E_λ and $\gamma_{\lambda c}$ it is very helpful to consider the case of a single level, i.e. $N_\lambda = 1$. In this case the $\boldsymbol{A}$ matrix is simply a number and the cross section is given by

$$\sigma_{cc'}(E) = \frac{\pi\omega}{k_c^2} \frac{\Gamma_{\lambda c}\Gamma_{\lambda c'}}{(E_\lambda + \Delta_\lambda - E)^2 + (\Gamma_\lambda/2)^2}, \tag{8}$$

where

$$\Gamma_{\lambda c} = 2\gamma_{\lambda c}^2 P_c, \tag{9}$$

$$\Delta_\lambda = -\sum_c \gamma_{\lambda c}^2 [S_c(E) - B_c], \text{ and} \tag{10}$$

$$\Gamma_\lambda = \sum_c \Gamma_{\lambda c}. \tag{11}$$

The partial and total widths $\Gamma_{\lambda c}$ and Γ_λ are known as *formal* widths. This formula for the cross section is very similar to the Breit-Wigner formula defined above by Eq. (1) (in fact the single-level $\boldsymbol{R}$-matrix formula is sometimes called the Breit-Wigner formula). The relationship between Eqs. (1) and (8) is simplest if the B_c are chosen such that $B_c = S_c(E_\lambda)$. The energy-dependent quantity Δ_λ can then be expanded in a Taylor series around $E = E_\lambda$:

$$\Delta_\lambda = (E - E_\lambda)\left(\frac{\Delta_c}{dE}\right)_{E_\lambda} + \ldots \tag{12}$$

$$= (E_\lambda - E)\sum_c \gamma_{\lambda c}^2 \left(\frac{dS_c}{dE}\right)_{E_\lambda} + \ldots \tag{13}$$

If we replace Δ_λ with the first non-zero term in the expansion, we have

$$(E_\lambda + \Delta_\lambda - E)^2 \approx (E_\lambda - E)^2 \left[1 + \sum_c \gamma_{\lambda c}^2 \left(\frac{dS_c}{dE}\right)_{E_\lambda}\right]^2. \tag{14}$$

We can now connect the $\boldsymbol{R}$-matrix parameters with the Breit-Wigner resonance parameters. If we identify

$$E_R = E_\lambda \quad \text{and} \tag{15}$$

$$\Gamma_{\lambda c}^o = \frac{\Gamma_{\lambda c}(E_\lambda)}{1 + \sum_c \gamma_{\lambda c}^2 \left(\frac{dS_c}{dE}\right)_{E_\lambda}} = \frac{2\gamma_{\lambda c}^2 P_c(E_\lambda)}{1 + \sum_c \gamma_{\lambda c}^2 \left(\frac{dS_c}{dE}\right)_{E_\lambda}}, \tag{16}$$

then the one-level $\boldsymbol{R}$-matrix formula [Eq. (8)] and the Breit-Wigner formula [Eq. (1)] are equivalent. The presence of the penetration factor in Eq. (9) gives rise to an energy dependence of the $\Gamma_{\lambda c}$ – hence the need for defining $\Gamma_{\lambda c}(E_\lambda)$ and $P_c(E_\lambda)$ in Eq. (16). Eqs. (15) and (16) can be considered as the definition of the resonance energy and partial widths for the case of a single-level $\boldsymbol{R}$-matrix (i.e. the *physical* parameters). These relationships are also discussed in Sec. XII.3(a) of LT. The resonance energy defined in this manner may differ from the peak of cross section excitation function due to the energy dependences of the penetration factors and k_c^2. Note also that the difference between formal and observed partial widths is often on the order of 30% – a large factor compared to typical experimental uncertainties.

V. GENERAL DEFINITION OF PHYSICAL RESONANCE PARAMETERS

When there is more than one level, a one-level approximation to the $\boldsymbol{R}$ matrix is very good when E is near the $\boldsymbol{R}$-matrix pole energy E_λ of a particular level [this approximation becomes exact when $E \to E_\lambda$; see Eq. (2)]. Eqs. (15) and (16) from the previous section thus define the physical parameters for that particular level. However, this approach requires defining the boundary condition constants such that $B_c = S_c(E_\lambda)$. The B_c can only be set once (they are after all "constants") – therefore we only have a simple relationship between $\boldsymbol{R}$-matrix parameters and physical parameters for one of the levels.

One approach to finding the physical parameters for the remaining levels is to change the B_c to values appropriate for another physical resonance energy. The $\boldsymbol{R}$-matrix parameters E_λ and $\gamma_{\lambda c}$ must then be re-determined – either by refitting data or by the transformation procedure discussed in the next section. This procedure must be then be repeated one level at a time until physical parameters have been found for all of the levels. Note also that this procedure is necessarily iterative – we want to set $B_c = S_c(E_\lambda)$ where E_λ is a physical resonance energy, but the physical resonance energy is one of the quantities we are seeking to determine. Fortunately an iterative approach starting with the $\boldsymbol{R}$-matrix pole energy converges quickly.

VI. HOW TO CHANGE THE B_c

An interesting feature of $\boldsymbol{R}$-matrix theory is that the collision matrix (i.e. the $\boldsymbol{U}$ matrix) is invariant under changes in the B_c, provided that the E_λ and $\gamma_{\lambda c}$ are suitably adjusted. This result remains true even for the case of finite N_λ [2]. The transformation is most easily described using matrix equations in level space. Let us consider the transformation $B_c \to B'_c$, $E_\lambda \to E'_\lambda$, and $\gamma_{\lambda c} \to \gamma'_{\lambda c}$. One first constructs the real and symmetric matrix $\boldsymbol{C}$ defined by

$$\boldsymbol{C} = \boldsymbol{e} - \sum_c \boldsymbol{\gamma}_c \boldsymbol{\gamma}_c^T (B'_c - B_c), \tag{17}$$

which is diagonalized by the orthogonal matrix $\boldsymbol{K}$ such that $\boldsymbol{D} = \boldsymbol{K}\boldsymbol{C}\boldsymbol{K}^T$, with $D_{\lambda\mu} = D_\lambda \delta_{\lambda\mu}$. Note that any real and symmetric matrix can be diagonalized by an orthogonal matrix; an orthogonal matrix $\boldsymbol{K}$ satisfies $\boldsymbol{K}^T = \boldsymbol{K}^{-1}$ (in other words $\boldsymbol{K}^T\boldsymbol{K} = \boldsymbol{K}\boldsymbol{K}^T = \boldsymbol{1}$). The necessary transformation of the $\boldsymbol{R}$-matrix parameters is then given by [2]

$$E'_\lambda = D_\lambda \tag{18}$$

and

$$\boldsymbol{\gamma}'_c = \boldsymbol{K}\boldsymbol{\gamma}_c. \tag{19}$$

By using Eq. (6) one can show that these transformations leave $\boldsymbol{\gamma}_{c'}^T \boldsymbol{A} \boldsymbol{\gamma}_c$ and hence $\boldsymbol{U}$ [see Eq. (7)] invariant.

The invariance of $\boldsymbol{U}$ for changes in B_c has several important consequences. I will highlight two of them: (1) mathematically-equivalent cross section fits should be attainable for any B_c choices and (2) we are free to choose B_c however we desire. We can exploit this freedom by adopting B_c values which are convenient [e.g. $B_c = S_c(E_\lambda)$]. This situation is somewhat analogous to that encountered in electromagnetism: transformations of the scalar and vector potentials (known as gauge transformations) can be carried out which leave the physical electric and magnetic fields invariant.

It is instructive to look at the transformation for the case of a single level. The matrix $\boldsymbol{C}$ consists of only a single dimension, i.e. a number. The resulting transformation is

$$E'_\lambda = E_\lambda - \sum_c \gamma_{\lambda c}^2 (B'_c - B_c) \tag{20}$$

$$\gamma'_{\lambda c} = \gamma_{\lambda c}. \tag{21}$$

It is straightforward to verify by substitution into the single-level formula [Eq. (8)] that the cross section is invariant.

VII. ANOTHER APPROACH TO PHYSICAL PARAMETERS

In this section I will describe another mathematically-equivalent method for extracting physical parameters from $\boldsymbol{R}$-matrix parameters. I begin by defining the real and symmetric matrix $\boldsymbol{\mathcal{E}}$:

$$\boldsymbol{\mathcal{E}} = \boldsymbol{e} - \sum_c \boldsymbol{\gamma}_c \boldsymbol{\gamma}_c^T [S_c(E) - B_c], \tag{22}$$

and consider the eigenvalue equation

$$\boldsymbol{\mathcal{E}} \boldsymbol{a}_i = \tilde{E}_i \boldsymbol{a}_i \tag{23}$$

where $\tilde{E}_i$ is the eigenvalue and $\boldsymbol{a}_i$ is the corresponding eigenvector. Note that $\boldsymbol{\mathcal{E}}$ is implicitly dependent upon $\tilde{E}_i$ through S_c, so the eigenvalue problem is nonlinear. We will assume for convenience that the eigenvectors are normalized so that $\boldsymbol{a}_i^T \boldsymbol{a}_i = 1$.

The eigenvalues $\tilde{E}_i$ are in fact just the physical resonance energies. A new set of reduced width parameters is defined via

$$\tilde{\gamma}_{ic} = \boldsymbol{a}_i^T \boldsymbol{\gamma}_c. \tag{24}$$

The $\tilde{\gamma}_{ic}$ are equal to the $\boldsymbol{R}$-matrix reduced widths when transformed to $B_c = S_c(\tilde{E}_i)$. The $\tilde{\gamma}_{ic}$ are sometimes called the "on-resonance" reduced-width amplitudes. The observed (physical) partial widths are thus given by

$$\Gamma_{ic}^o = \frac{2P_c(\tilde{E}_i)\tilde{\gamma}_{ic}^2}{1 + \sum_c \tilde{\gamma}_{ic}^2 \left(\frac{dS_c}{dE}\right)_{\tilde{E}_i}}. \tag{25}$$

One advantage of this formulation is that the solution of the eigenvalue problem gives immediately all of the physical resonance energies and partial widths.

However one defines the physical resonance parameters, it is desirable that they be independent of the boundary condition constants and channel radii of $\boldsymbol{R}$-matrix theory. The above approach is completely independent of the B_c and approximately independent of the channel radii. Additional details and the justification for this approach are given by Brune [3]. The "on-resonance" reduced-width amplitudes defined here are also disused by Barker [4] and by Angulo and Descouvemont [5].

I would like to mention in passing another similar eigenvalue equation. If we had instead defined

$$\boldsymbol{\mathcal{E}} = \boldsymbol{e} - \sum_c \boldsymbol{\gamma}_c \boldsymbol{\gamma}_c^T [S_c(E) + \mathrm{i}P_c(E) - B_c], \tag{26}$$

we would have the eigenvalue equation introduced by Hale, Brown, and Jarmie [6]. The eigenvalues of this equation are complex and correspond to complex poles of the $\boldsymbol{A}$ and $\boldsymbol{U}$ matrices (the latter is also known as the scattering matrix). As discussed in Refs. [6, 7] this approach to defining resonance energies and widths may be theoretically preferable. However, there are significant complexities associated with calculating the Coulomb wavefunctions for complex momenta. We will not discuss this approach further here.

VIII. CONVERTING PHYSICAL PARAMETERS TO *R*-MATRIX PARAMETERS

Next we will investigate how to translate physical parameters into $\boldsymbol{R}$-matrix parameters [3]. It is often desirable to incorporate information from a level compilation or shell model calculation into an $\boldsymbol{R}$-matrix parameterization. Values for Γ_{ic}^o must first be converted into reduced widths $\tilde{\gamma}_{ic}$ using Eq. (25). This procedure requires the straightforward solution of a system of linear equations for $\tilde{\gamma}_{ic}^2$; the only complication is that the *signs* of reduced-width amplitudes cannot be determined. Relative signs can be determined via an $\boldsymbol{R}$-matrix calculation (which automatically includes interference effects between levels) and a comparison to experimental data.

With the physical resonance energies $\tilde{E}_i$ and "on-resonance" reduced-width amplitudes $\tilde{\gamma}_{ic}$ in hand we can determine $\boldsymbol{R}$-matrix parameters. We proceed by defining the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$:

$$[\boldsymbol{M}]_{ij} \equiv \begin{cases} 1 & i = j \\ -\sum_c \tilde{\gamma}_{ic}\tilde{\gamma}_{jc} \frac{S_{ic} - S_{jc}}{\tilde{E}_i - \tilde{E}_j} & i \neq j \end{cases} \tag{27}$$

and

$$[\boldsymbol{N}]_{ij} \equiv \begin{cases} \tilde{E}_i + \sum_c \tilde{\gamma}_{ic}^2 (S_{ic} - B_c) & i = j \\ \sum_c \tilde{\gamma}_{ic}\tilde{\gamma}_{jc} \left(\frac{\tilde{E}_i S_{jc} - \tilde{E}_j S_{ic}}{\tilde{E}_i - \tilde{E}_j} - B_c \right) & i \neq j \end{cases}, \tag{28}$$

where S_{ic} denotes $S_c(\tilde{E}_i)$. Note that the construction of $\boldsymbol{N}$ requires the adoption of specific B_c values. The eigenvalue equation

$$\boldsymbol{N}\boldsymbol{b}_\lambda = E_\lambda \boldsymbol{M}\boldsymbol{b}_\lambda. \tag{29}$$

now holds the key for transforming from the $\tilde{E}_i$-$\tilde{\gamma}_{ic}$ representation to the standard $\boldsymbol{R}$-matrix parameters E_λ and $\gamma_{\lambda c}$. The eigenvalues are the $\boldsymbol{R}$-matrix pole energies E_λ and the eigenvectors $\boldsymbol{b}_\lambda$ yield the $\boldsymbol{R}$-matrix reduced-width amplitudes:

$$\gamma_{\lambda c} = \boldsymbol{b}_\lambda^T \tilde{\boldsymbol{\gamma}}_c. \tag{30}$$

The matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ are real, symmetric, and energy-independent. The solution to this type of eigenvalue problem is discussed in Sec. 8.7.2 of Ref. [8]; I have utilized the LAPACK [9] routine DSYGV.

The eigenvectors $\boldsymbol{b}_\lambda$ can be arranged into a square matrix $\boldsymbol{b}$ which simultaneously diagonalizes $\boldsymbol{M}$ and $\boldsymbol{N}$:

$$\boldsymbol{b}^T \boldsymbol{M}\boldsymbol{b} = \boldsymbol{1} \tag{31}$$

and

$$\boldsymbol{b}^T \boldsymbol{N}\boldsymbol{b} = \boldsymbol{e}. \tag{32}$$

We can also re-write Eq. (30) as

$$\boldsymbol{\gamma}_c = \boldsymbol{b}^T \tilde{\boldsymbol{\gamma}}_c. \tag{33}$$

IX. WORKING DIRECTLY WITH PHYSICAL PARAMETERS

It turns to be possible to carry out $\boldsymbol{R}$-matrix calculations directly from the physical parameters [3]. This result can be achieved by defining a new (alternative) level matrix $\tilde{\boldsymbol{A}}$ from the physical resonance energies $\tilde{E}_i$ and "on-resonance" reduced-width amplitudes $\tilde{\gamma}_{ic}$:

$$\begin{aligned} [\tilde{\boldsymbol{A}}^{-1}]_{ij} &= (\tilde{E}_i - E)\delta_{ij} - \sum_c \tilde{\gamma}_{ic}\tilde{\gamma}_{jc}(S_c + \mathrm{i}P_c) \\ &+ \sum_c \begin{cases} \tilde{\gamma}_{ic}^2 S_{ic} & i = j \\ \tilde{\gamma}_{ic}\tilde{\gamma}_{jc} \frac{S_{ic}(E - \tilde{E}_j) - S_{jc}(E - \tilde{E}_i)}{\tilde{E}_i - \tilde{E}_j} & i \neq j \end{cases}. \end{aligned} \tag{34}$$

This matrix is then used with the $\tilde{\boldsymbol{\gamma}}_c$ to calculate the the collision matrix $\boldsymbol{U}$:

$$U_{c'c} = \Omega_{c'}\Omega_c \left[\delta_{c'c} + 2\mathrm{i}(P_{c'}P_c)^{1/2} \tilde{\boldsymbol{\gamma}}_{c'}^T \tilde{\boldsymbol{A}} \tilde{\boldsymbol{\gamma}}_c \right]. \tag{35}$$

This method of calculating $\boldsymbol{U}$ is mathematically equivalent to first transforming the physical $\tilde{E}_i$-$\tilde{\gamma}_{ic}$ parameters to $\boldsymbol{R}$-matrix parameters E_λ-$\gamma_{\lambda c}$ and then calculating $\boldsymbol{U}$ using standard $\boldsymbol{R}$-matrix equations. Note that in this approach the boundary-condition constants B_c never need to be defined at all. One application of this approach is the possibility of using the physical parameters directly as fit parameters.

X. AN EXAMPLE

At this point it is instructive to introduce a concrete example. The top section of Table I shows a simple well-documented set of standard $\boldsymbol{R}$-matrix parameters taken from Azuma *et al.* [10]. This analysis concerns the low-lying $J^\pi = 1^-$ states of ^{16}O reacting through channels $^{12}\mathrm{C} + \alpha$ ($c = \alpha$, $l_\alpha = 1$) and $^{16}\mathrm{O} + \gamma$ ($c = \gamma$, $E1$ multipolarity). The zero of the energy scale is at the $^{12}\mathrm{C} + \alpha$ threshold. Note that the gamma-ray channel is treated perturbatively and does enter in the calculation of the $\boldsymbol{A}$ matrix.

TABLE I: Standard $\boldsymbol{R}$-matrix parameters from Table III of Ref. [10] which describe $J^\pi = 1^-$ reactions in the ^{16}O system, and the physical parameters derived from them. The channel radius is 6.5 fm and the boundary condition constant is chosen according to $B_c = S_c(E_1)$. The β-decay feeding amplitudes $\mathcal{B}_\lambda$ are equivalent to the quantities $A_{\lambda 1}\gamma_{\lambda 1}^{-1} N_\alpha^{-1/2}$ of Ref. [10].

λ	1	2	3
E_λ (MeV)	-0.0451	2.845	11.71
$\gamma_{\lambda\alpha}$ (MeV$^{1/2}$)	0.0793	0.330	1.017
$\gamma_{\lambda\gamma}$ (MeV^{-1})	8.76×10^{-6}	-2.44×10^{-6}	-2.82×10^{-6}
$\mathcal{B}_\lambda$	1.194	0.558	-0.629
$\tilde{E}_\lambda$ (MeV)	-0.0451	2.400	8.00
$\tilde{\gamma}_{\lambda\alpha}$ (MeV$^{1/2}$)	0.0793	0.471	0.912
$\tilde{\gamma}_{\lambda\gamma}$ (MeV^{-1})	8.76×10^{-6}	-3.20×10^{-6}	-2.50×10^{-6}
$\tilde{\mathcal{B}}_\lambda$	1.194	0.408	-0.781
$1 + \tilde{\gamma}_{\lambda\alpha}^2 \frac{dS_\alpha}{dE}(\tilde{E}_\lambda)$	1.0050	1.2034	1.0220
Γ_α^o (MeV)	–	0.359	9.041
Γ_γ^o (meV)	55.1	8.64	54.2

TABLE II: Elements $[\boldsymbol{b}]_{ij}$ of the transformation matrix corresponding to the parameters of Table I.

$i \backslash j$	1	2	3
1	1.000	0.0373	0.0446
2	0.000	0.9781	0.2281
3	0.000	-0.1466	0.9933

The bottom section of Table I shows physical resonance energies $\tilde{E}_i$ and the "on-resonance" reduced-width amplitudes $\tilde{\gamma}_{\lambda c}$ and β-decay feeding factors $\mathcal{B}_i$. In addition the physical partial widths Γ_{ic}^o are given along with values of $[1 + \tilde{\gamma}_{i\alpha}^2 \frac{dS_\alpha}{dE}(\tilde{E}_i)]$ needed to related formal and physical widths.

This example is also useful for demonstrating some of the other concepts discussed here. For example one can convert from the $\tilde{E}_i$-$\tilde{\gamma}_{ic}$ representation back to the standard $\boldsymbol{R}$-matrix E_λ-$\gamma_{\lambda c}$ representation. In Table II we show the elements of the transformation matrix $\boldsymbol{b}$ [see Eqs. (31-33)] needed to carry the transformation back to standard $\boldsymbol{R}$-matrix parameters. One can also verify that the various approaches for calculating the collision matrix $\boldsymbol{U}$ [e.g. Eqs. (7) and (35)] yield the same result.

XI. SUMMARY

Several aspects of the relationship between $\boldsymbol{R}$-matrix parameters and physical resonance parameters have been discussed. First it is necessary to establish how physical resonance energies and partial widths will be defined. By comparing the one-level $\boldsymbol{R}$-matrix formula for the cross section to the Breit-Wigner formula it is possible to define a relationship between $\boldsymbol{R}$-matrix parameters and physical resonance parameters. For the multi-level case the extraction of physical parameters is more complicated but can be achieved by the refitting data with different B_c values, carrying out B_c transformations, or by solving an eigenvalue equation. We next investigated the reverse procedure, i.e. transforming physical resonance parameters into $\boldsymbol{R}$-matrix parameters, which is useful for incorporating information from other sources into an $\boldsymbol{R}$-matrix fit. A method for calculating observables directly from the "on-resonance" $\boldsymbol{R}$-matrix parameters was then discussed. Finally we presented an example case which is useful for demonstrating the ideas which have been presented here.

[1] A. M. Lane and R. G. Thomas, Rev. Mod. Phys. **30**, 257 (1958).
[2] F. C. Barker, Aust. J. Phys. **25**, 341 (1972).
[3] C. R. Brune, Phys. Rev. C **66**, 044611 (2002).
[4] F. C. Barker, Aust. J. Phys **24**, 777 (1971). The quantities $E_{\lambda l}^{(\lambda)}$ and $\gamma_{\lambda l}^{(\lambda)}$ of Ref. [4] correspond to $\tilde{E}_\lambda$ and $\tilde{\gamma}_{\lambda c}$ in the present work.

[5] C. Angulo and P. Descouvemont, Phys. Rev. C **61**, 064611 (2000). We have utilized a notation somewhat different from Ref. [5], in order that we could retain the well-established notation of Lane and Thomas [1]. The quantities E^r_λ and γ_λ of Ref. [5] correspond to $\tilde{E}_\lambda$ and $\tilde{\gamma}_{\lambda c}/\sqrt{1+\sum_c \tilde{\gamma}^2_{\lambda c}\left(\frac{dS_c}{dE}\right)_{\tilde{E}_\lambda}}$ in the present work. Note also that the procedure of Ref. [5] is approximate and only considers the single-channel case.
[6] G. M. Hale, R. E. Brown, and N. Jarmie, Phys. Rev. Lett. **59**, 763 (1987).
[7] D. R. Tilley, C. M. Cheves, J. L. Godwin, G. M. Hale, H. M. Hofmann, J. H. Kelley, C. G. Sheu, and H. R. Weller, Nucl. Phys. **A708**, 3 (2002).
[8] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd Ed. (Johns Hopkins University Press, Baltimore, 1996).
[9] E. Anderson Z. Bai, C. Bischof, S. Blackford, J. Demmel, J. Dongarra, J. Du Croz, A. Greenbaum, S. Hammarling, A. McKenney, D. Sorensen, *LAPACK Users' Guide*, 3rd Ed. (Society for Industrial and Applied Mathematics, Philadelphia, 1999).
[10] R. E. Azuma, L. Buchmann, F. C. Barker, C. A. Barnes, J. M. D'Auria, M. Dombsky, U. Giesen, K. P. Jackson, J. D. King, R. G. Korteling, P. McNeely, J. Powell, G. Roy, J. Vincent, T. R. Wang, S. S. M. Wong, and P. R. Wrean, Phys. Rev. C **50**, 1194 (1994).