Formal and Physical R-Matrix Parameters

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R-Matrix Method

- Exact implimentaton of quantum-mechanical symmetries and conservation laws (Unitarity)
- Treats long-ranged Coulomb potential explicitly
- Wavefunctions are expanded in terms of unknown basis functions
- Energy eigenvalues and the matrix elements of basis functions are adjustable parameters
- A wide range of physical observables can be fitted (e.g. cross sections, E_x, Γ_x,…)
- The fit can then be used to determine unmeasured observables
- Major Approximation: TRUNCATION (levels / channels)

R-Matrix Parameters

- λ – level label
- c – channel label (e.g. α–particle, γ-ray)
- E_λ – level energy
- $\gamma_{\lambda c}$ – reduced width amplitude
- B_c – boundary condition constants (related to “level shift”)

How does one define a resonance energy?

- Peaks of excitation function ?
- Phase shift = $\pi/2$? (for elastic scattering)
- Complex poles of the S-matrix (or U-matrix) ?
- …
- Bottom line: There is no "right" answer, be careful about "apples and oranges".
- I will describe how we can do this in the R-matrix formalism.

Breit-Wigner Formula

$$\sigma_{cc'} = \frac{\pi\omega}{k^2}\frac{\Gamma_c^o\Gamma_{c'}^o}{(E_R - E)^2 + (\Gamma^o/2)^2}$$

$$\Gamma^o = \sum_c \Gamma_c^o$$

- General QM result (from atomic physics to Z_0 bosons)
- Γ^0_c are observed partial widths
- E_R and Γ^0_c are considered “Physical Parameters”
- Physical Parameters should be independent of boundary condition constants (B_c) and channel radius (a)

1-Level R-Matrix Formula

$$\sigma_{cc'} = \frac{\pi\omega}{k^2}\frac{\Gamma_{\lambda c}\Gamma_{\lambda c'}}{(E_\lambda + \Delta_\lambda - E)^2 + (\Gamma_\lambda/2)^2}$$

$$\begin{aligned}
\Gamma_{\lambda c} &= 2\gamma_{\lambda c}^2 P_c \\
\Delta_\lambda &= -\sum_c \gamma_{\lambda c}^2 [S_c(E) - B_c] \\
\Gamma_\lambda &= \sum_c \Gamma_{\lambda c}
\end{aligned}$$

- Very similar to general Breit-Wigner formula
- $\Gamma_{\lambda c}$ are formal partial widths

Make it look like the B-W Formula:

$$\sigma_{cc'} = \frac{\pi\omega}{k^2} \frac{\Gamma_{\lambda c}\Gamma_{\lambda c'}}{(E_\lambda + \Delta_\lambda - E)^2 + (\Gamma_\lambda/2)^2}$$

1-level R-matrix

$$\sigma_{cc'} = \frac{\pi\omega}{k^2} \frac{\Gamma^o_c\Gamma^o_{c'}}{(E_R - E)^2 + (\Gamma^o/2)^2}$$

$$\Gamma^o = \sum_c \Gamma^o_c$$

"standard" Breit-Wigner

$$\begin{aligned}
E_\lambda &= E_R \\
B_c &= S_c(E_\lambda) \\
\Delta_\lambda &= (E - E_\lambda)\left(\frac{\Delta_c}{dE}\right)_{E_\lambda} + \ldots \\
&= (E_\lambda - E)\sum_c \gamma^2_{\lambda c}\left(\frac{dS_c}{dE}\right)_{E_\lambda} + \ldots \\
(E_\lambda + \Delta_\lambda - E)^2 &\approx (E_\lambda - E)^2\left[1 + \sum_c \gamma^2_{\lambda c}\left(\frac{dS_c}{dE}\right)_{E_\lambda}\right]^2 \\
\Gamma^o_{\lambda c} &= \frac{\Gamma_{\lambda c}}{1 + \Sigma_c \gamma^2_{\lambda c}\left(\frac{dS_c}{dE}\right)_{E_\lambda}}
\end{aligned}$$

What if we have more than one level?

$$R_{cc'} = \sum_{\lambda} \frac{\gamma_{\lambda c}\gamma_{\lambda c'}}{E_\lambda - E}$$

- 1-level approximation very good when E is near E_λ.
- Same procedure can be used.
- But B_c can only be set once [recall $B_c = S_c(E_\lambda)$]
- $\Rightarrow$ Simple relation to physical parameters only for one level

Can we change the B_c?

- Yes !
- If $B_c \Rightarrow B_c$' then if

$$
\begin{aligned}
[\boldsymbol{C}]_{\lambda\mu} &= E_\lambda \delta_{\lambda\mu} - \sum_c \gamma_{\lambda c}\gamma_{\mu c}(B_c' - B_c) \\
\boldsymbol{D} &= \boldsymbol{K}\boldsymbol{C}\boldsymbol{K}^T \\
\boldsymbol{D}_{\lambda\mu} &= D_\lambda \delta_{\lambda\mu}
\end{aligned}
$$

$$
\boldsymbol{\gamma}_c = \begin{pmatrix} \gamma_{1c} \\ \gamma_{2c} \\ \gamma_{3c} \\ \ldots \end{pmatrix}
$$

and E_λ' and $\gamma_{\lambda c}$' are given by

$$
\begin{aligned}
E_\lambda' &= D_\lambda \\
\boldsymbol{\gamma}_c' &= \boldsymbol{K}\boldsymbol{\gamma}_c
\end{aligned}
$$

THEN the U matrix (i.e. cross sections, etc…) are not changed!

- F.C. Barker Aust. J. Phys 25, 341 (1972).
- In practice you could also re-fit the data with different B_c.

Definition of the A Matrix

$$[\boldsymbol{A}^{-1}]_{\lambda\mu} = (E_\lambda - E)\delta_{\lambda\mu} - \sum_c \gamma_{\lambda c}\gamma_{\mu c}(S_c + \mathrm{i}P_c - B_c)$$

- The A matrix determines the U matrix
- The U matrix determines observables (cross sections, etc…)

$$U_{c'c} = \Omega_{c'}\Omega_c\left[\delta_{c'c} + 2\mathrm{i}(P_{c'}P_c)^{1/2}\boldsymbol{\gamma}_{c'}^T\boldsymbol{A}\boldsymbol{\gamma}_c\right]$$

Extracting Physical Parameters is Iterative

Another approach: Solve Eigenvalue Equation

$$[\boldsymbol{\mathcal{E}}]_{\lambda\mu} = E_\lambda \delta_{\lambda\mu} - \sum_c \gamma_{\lambda c}\gamma_{\mu c}(S_c - B_c)$$

$$\mathcal{E}\boldsymbol{a}_i = \tilde{E}_i \boldsymbol{a}_i$$

Non-linear!

- Eigenvalues are the resonance energies
- Eigenvectors yield the physical partial widths

Note: G.M. Hale studies the complex eigenvalues of this equation:

$$[\boldsymbol{\mathcal{E}}]_{\lambda\mu} = E_\lambda \delta_{\lambda\mu} - \sum_c \gamma_{\lambda c}\gamma_{\mu c}(S_c + \mathrm{i}P_c - B_c)$$

Physical Parameters to R-Matrix Parameters

$$M_{ij} = \begin{cases} 1 & i = j \\ -\Sigma_c \tilde{\gamma}_{ic}\tilde{\gamma}_{jc}\frac{S_{ic}-S_{jc}}{\tilde{E}_i-\tilde{E}_j} & i \neq j \end{cases}$$
$$N_{ij} = \begin{cases} \tilde{E}_i + \Sigma_c \tilde{\gamma}_{ic}^2 (S_{ic} - B_c) & i = j \\ \Sigma_c \tilde{\gamma}_{ic}\tilde{\gamma}_{jc}\left(\frac{\tilde{E}_i S_{jc}-\tilde{E}_j S_{ic}}{\tilde{E}_i-\tilde{E}_j} - B_c\right) & i \neq j \end{cases}$$

$$\boldsymbol{N}\boldsymbol{b}_\lambda = E_\lambda \boldsymbol{M}\boldsymbol{b}_\lambda$$

- Another eigenvalue equation
- The eigenvalues are E_λ
- The eigenvectors can be arranged into a matrix b which diagonalizes M and N, and also yields γ_c

$$\boldsymbol{b}^T\boldsymbol{M}\boldsymbol{b} = \boldsymbol{1} \quad \text{and} \quad \boldsymbol{b}^T\boldsymbol{N}\boldsymbol{b} = \boldsymbol{e}$$

$$\boldsymbol{\gamma}_c = \boldsymbol{b}^T\tilde{\boldsymbol{\gamma}}_c$$

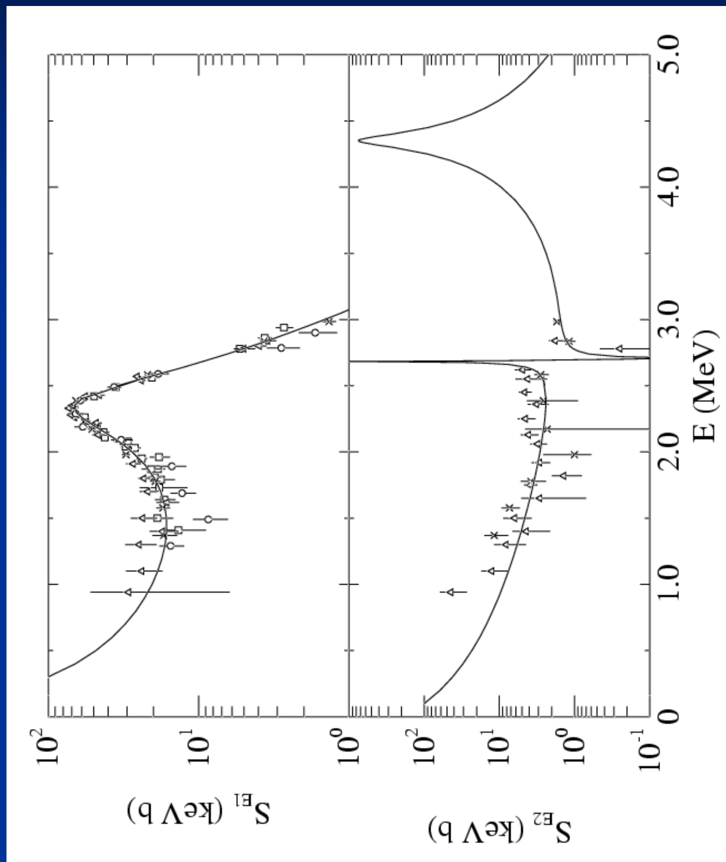
Working Directly with Physical Parameters

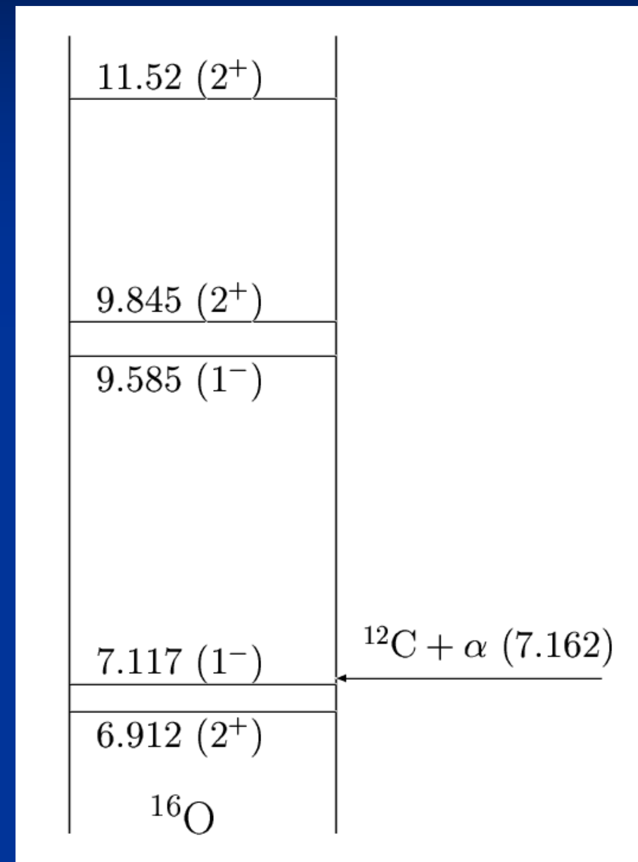
$$[\tilde{\boldsymbol{A}}^{-1}]_{ij} = (\tilde{E}_i - E)\delta_{ij} - \sum_c \tilde{\gamma}_{ic}\tilde{\gamma}_{jc}(S_c + \mathrm{i}P_c) + \sum_c \begin{cases} \tilde{\gamma}_{ic}^2 S_{ic} & i = j \\ \tilde{\gamma}_{ic}\tilde{\gamma}_{jc}\frac{S_{ic}(E-\tilde{E}_j) - S_{jc}(E-\tilde{E}_i)}{\tilde{E}_i - \tilde{E}_j} & i \neq j \end{cases}$$

$$U_{c'c} = \Omega_{c'}\Omega_c \left[\delta_{c'c} + 2\mathrm{i}(P_{c'}P_c)^{1/2}\tilde{\boldsymbol{\gamma}}_{c'}^T \tilde{\mathbf{A}}\tilde{\boldsymbol{\gamma}}_c\right]$$

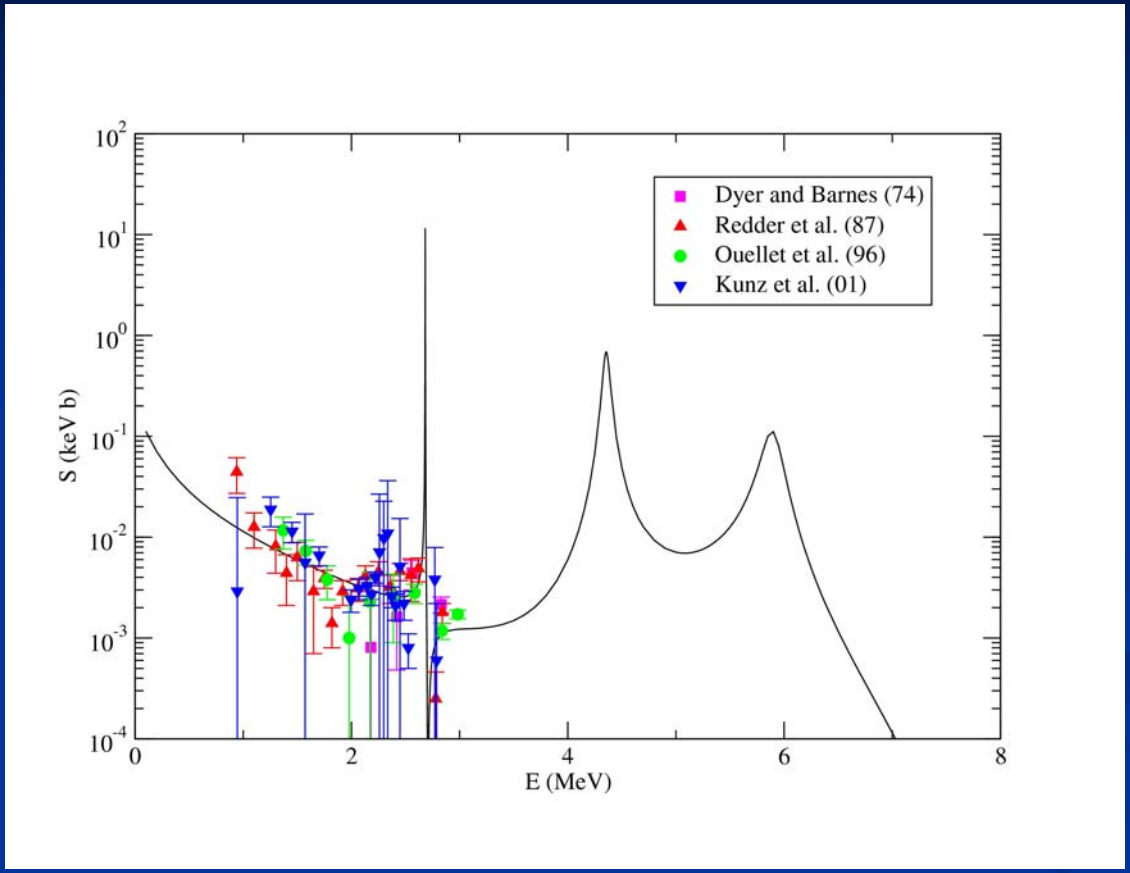
- Definition of the A matrix in terms of physical parameters
- Mathematically equivalent to Lane and Thomas (i.e. same U)
- See C.R. Brune, Phys. Rev. C 66, 044611 (2002).

1^- and 2^+ states of ^{16}O

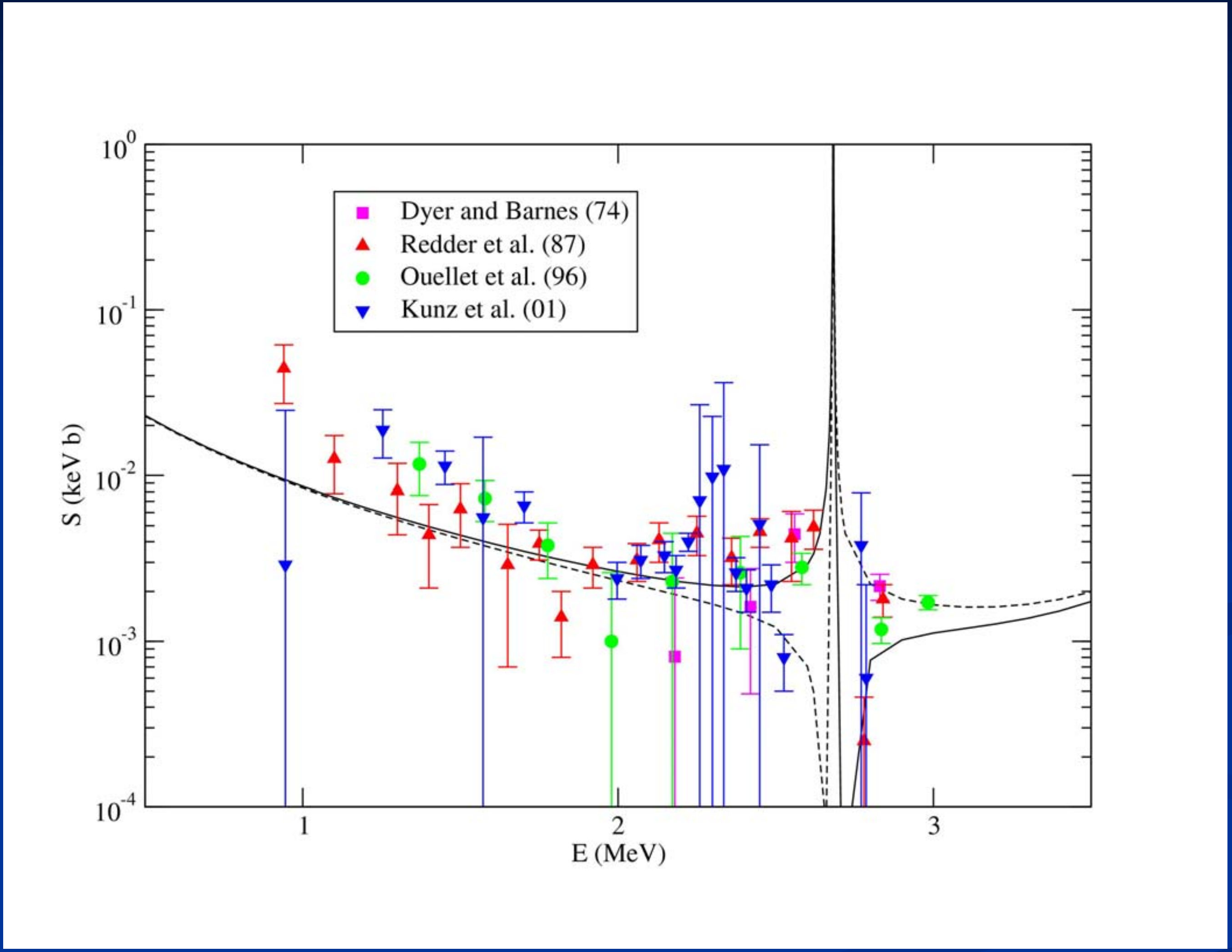




E2 Ground-State Cross Section



Interference near the 2.68-MeV Resonance (E2)



Summary

- Definition of physical parameters ($E_{R,}\ \Gamma_c^o$)
- How to get physical parameters from R-matrix parameters
- How to get R-matrix parameters from physical paratmeters