Topics in Nuclear Astrophysics II

Stellar Reaction Rates

- definition of a reaction rate
- Gamow window
- lifetimes of isotopes at stellar conditions
- nuclear energy production rate
- introduction to network simulations
- input parameters for calculating a reaction rate
- uncertainties in stellar reaction rates

Reaction Rate Definition

For a given relative velocity v with projectile number density n_p

$$\lambda = \sigma \cdot n_p \cdot v \left[s^{-1} \right]$$

reaction/target particle

energy/temperature dependent decay constant λ

$$R = \sigma \cdot n_p \cdot v \cdot n_T \cdot V \left[s^{-1} \right]$$

reaction rate in volume V

Reaction Rate in Stellar Environment

reaction rate per second and cm^3:

$$r = n_p \cdot n_T \cdot \sigma \cdot v$$

Reaction rate for particles with velocity distribution $\Phi(v)$

$$r = \frac{1}{1+\delta_{pT}} n_p \cdot n_T \cdot \int \sigma \cdot v \cdot \Phi(v) \cdot dv$$

Accounting for reactions
Between identical particles

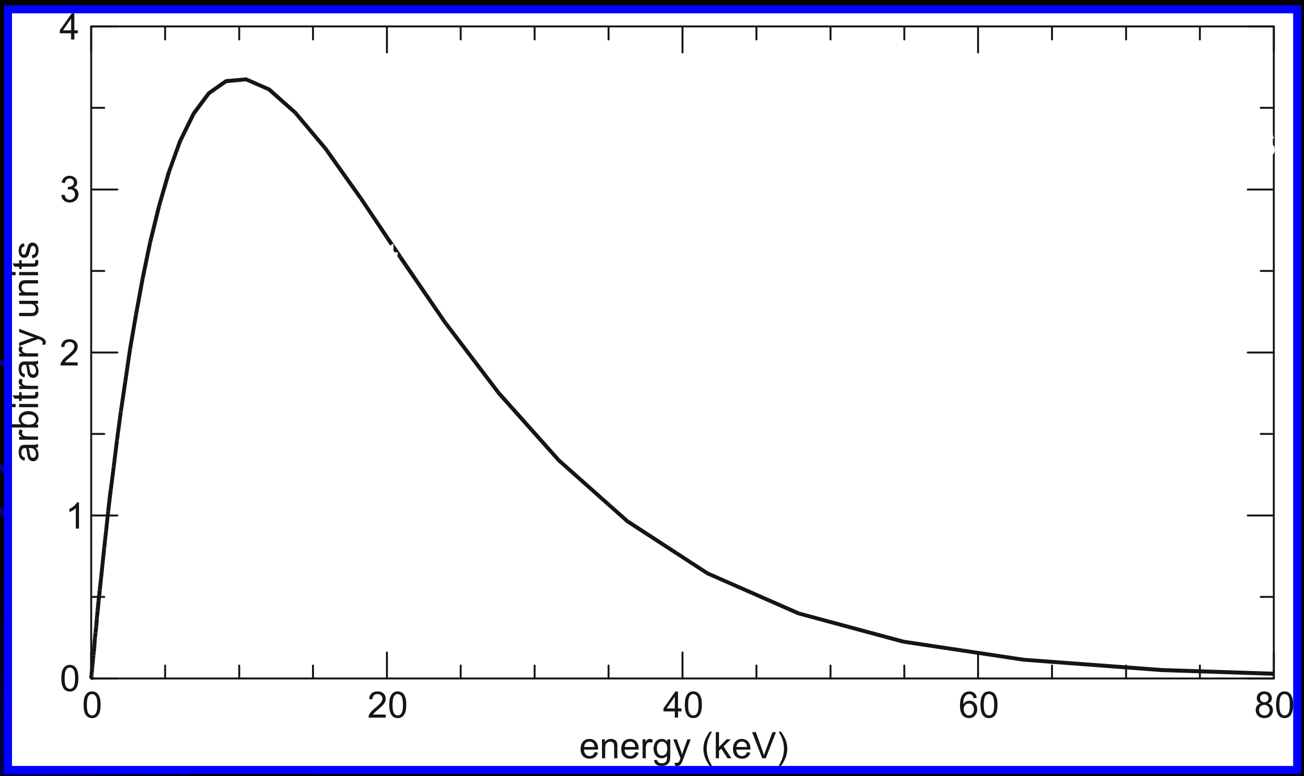
Maxwell Boltzmann Distribution

In stellar material of temperature T particles follow ideal gas law

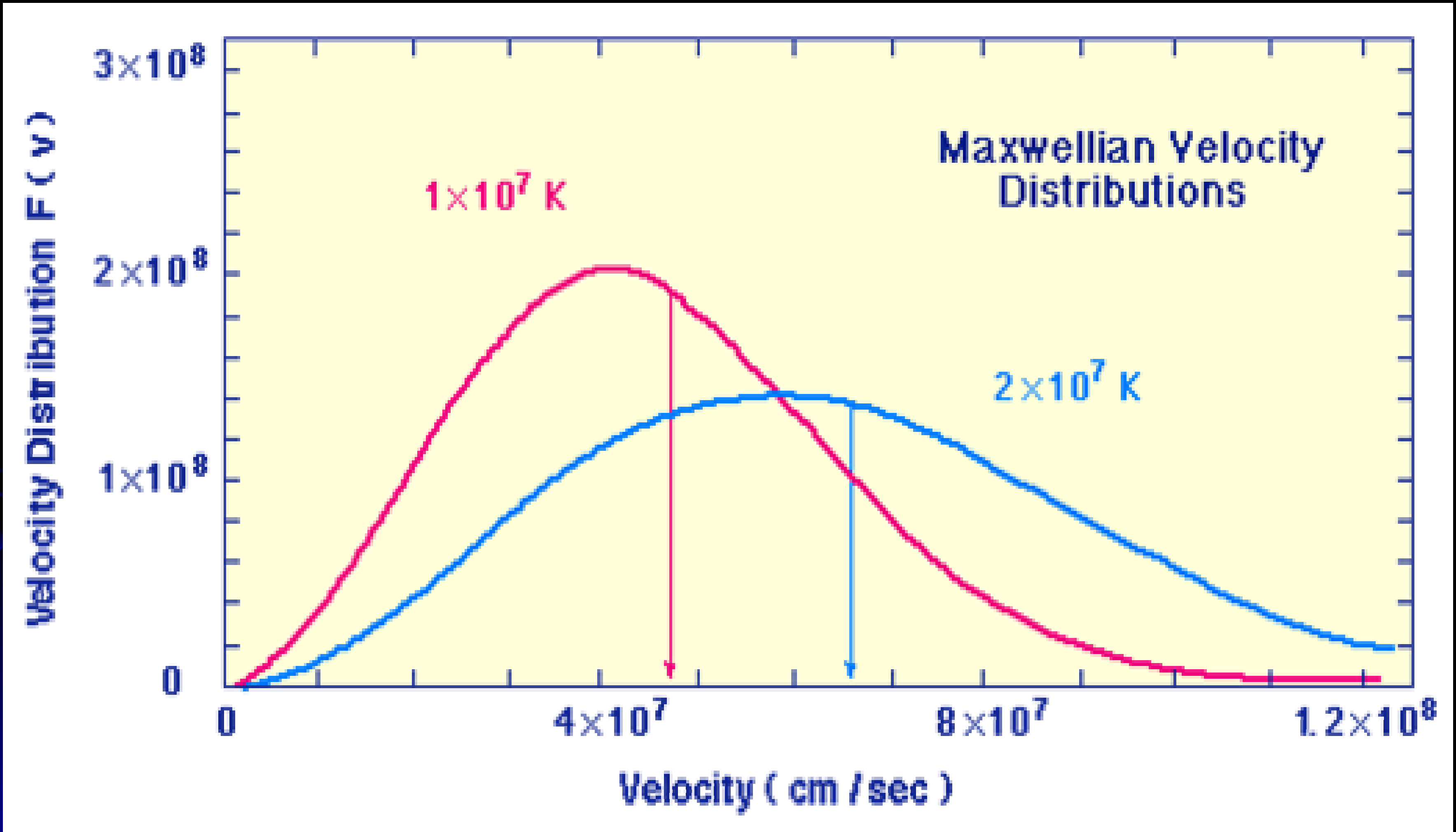
$$\Phi(v) = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} v^2 \, e^{-\frac{m v^2}{2kT}}$$

with

$$\int \Phi(v) dv = 1$$

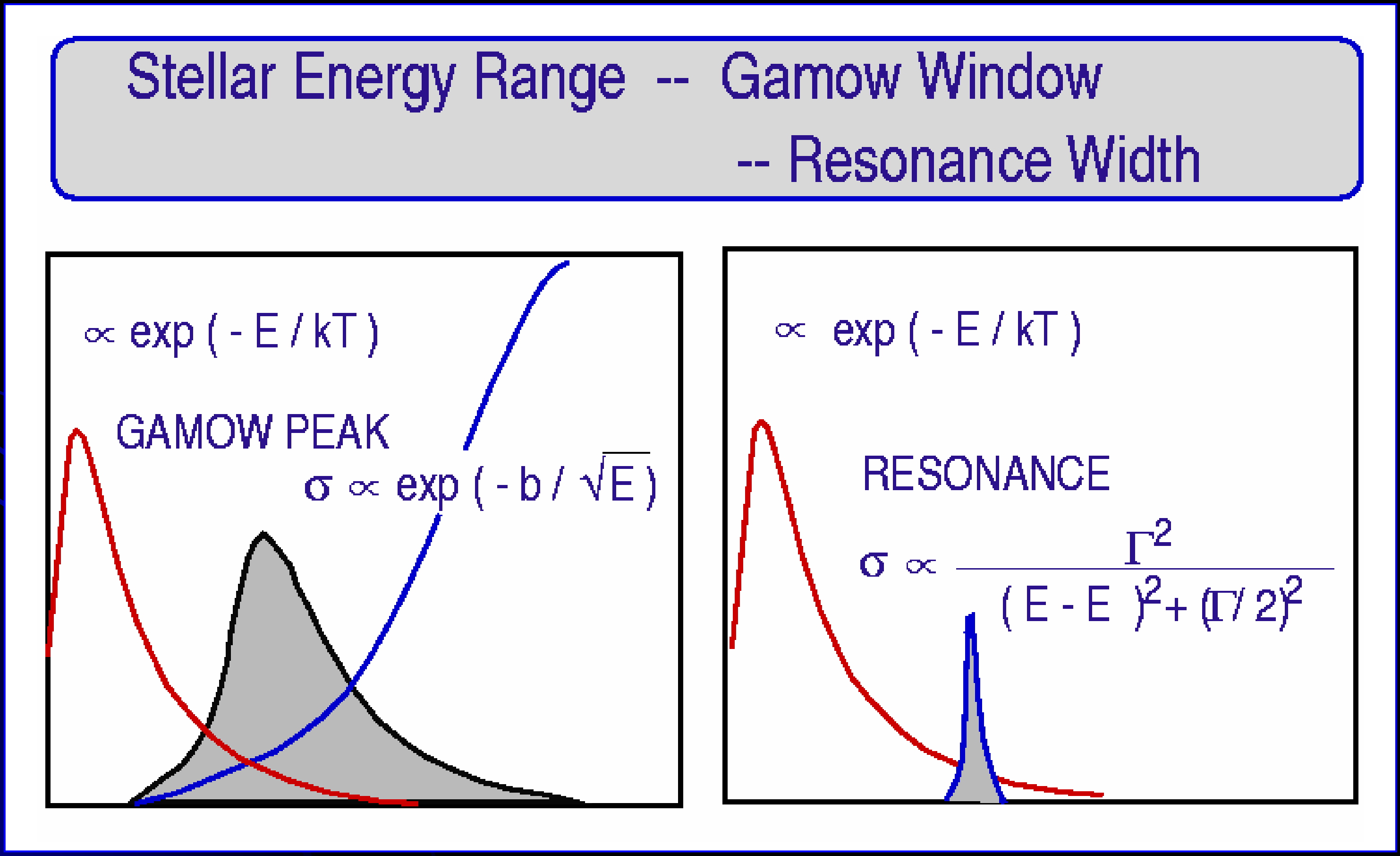


Temperature in Stars



Gamow Window & Reaction Rate

The Gamow window is the energy range for which the cross section needs to be experimentally or theoretically known!



The Gamow Range of Stellar Burning

The **Gamow window** or the range of relevant cross section is calculated:

$$E_0 = \left(\frac{bkT}{2}\right)^{3/2} = 0.122 \cdot \left(Z_1^2 Z_2^2 A\right)^{1/3} T_9^{2/3} \text{ MeV}$$

$$\Delta E = \frac{4}{\sqrt{3}} \sqrt{E_0 kT} = 0.2368 \cdot \left(Z_1^2 Z_2^2 A\right)^{1/6} T_9^{5/6} \text{ MeV}$$

with A “reduced mass number” and T_9 the temperature in GK

The Gamow peak for $^{12}C(p,\gamma)^{13}N$

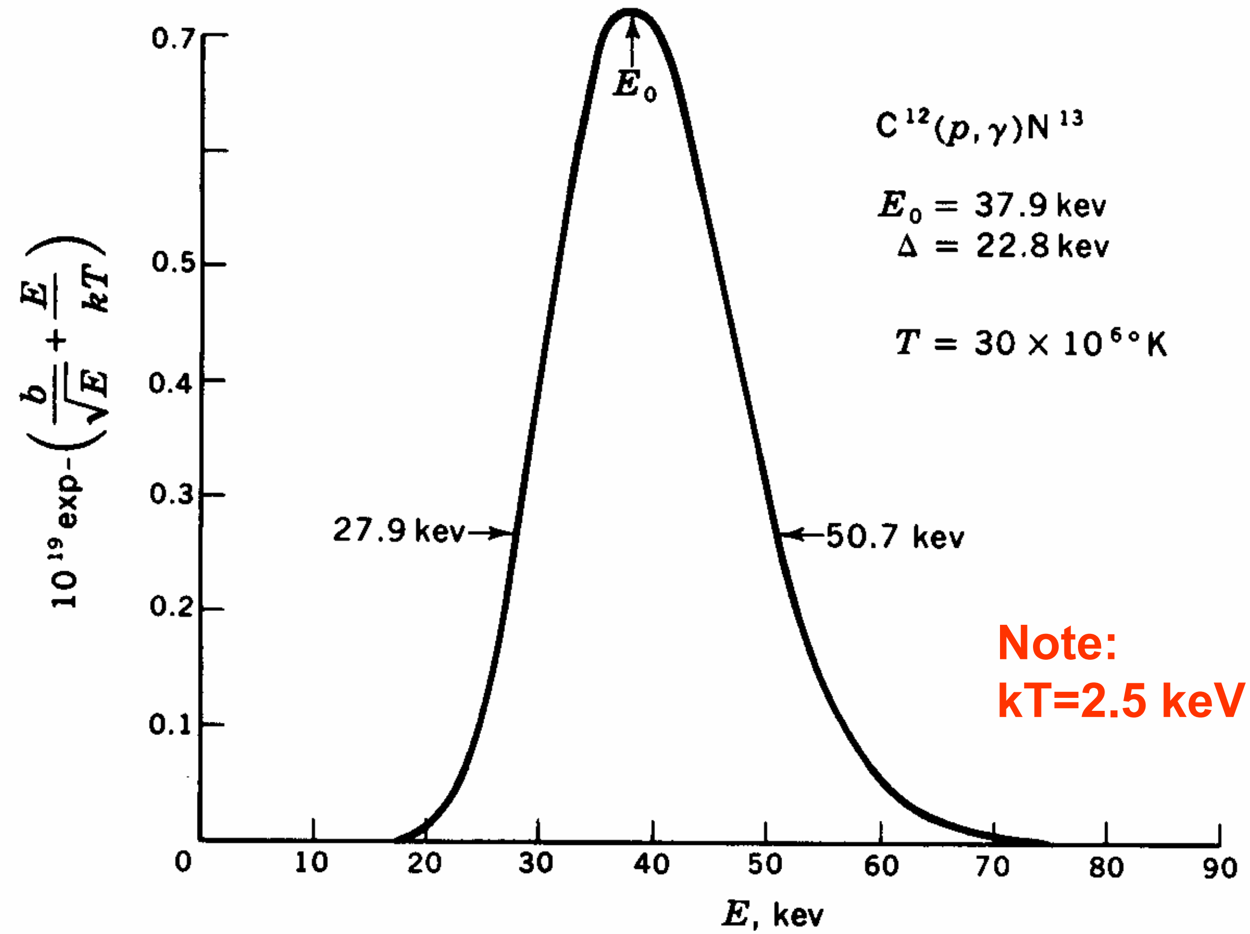
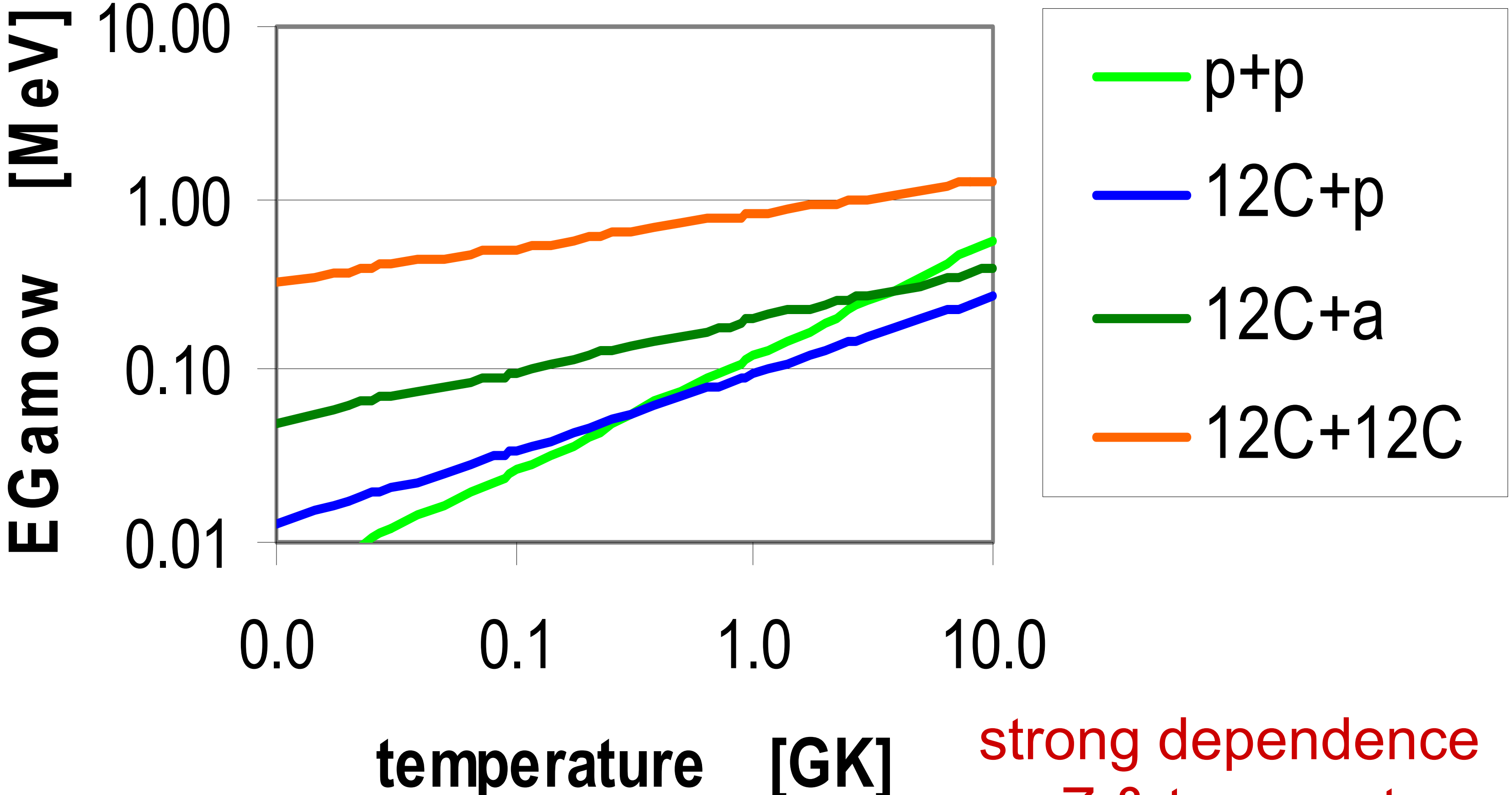


Fig. 4-7 The Gamow peak for the reaction $C^{12}(p,\gamma)N^{13}$ at $T = 30 \times 10^6$ °K. The curve is actually somewhat asymmetric about E_0, but it is nonetheless adequately approximated by a gaussian.

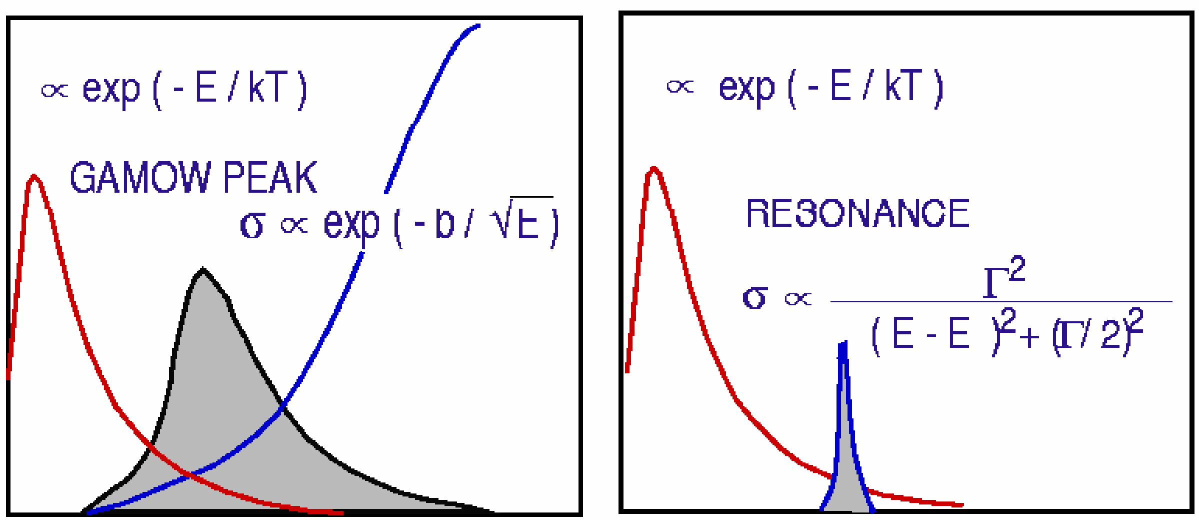
Note: kT=2.5 keV !

Examples of Gamow window energies



strong dependence on Z & temperature

Gamow-Range & Reaction Rate



Nonresonant Reaction Contributions

$$N_A < \sigma v > \propto T^{-3/2} \int \sigma E \exp (- E / kT) \, dE$$

Resonant Reaction Rate

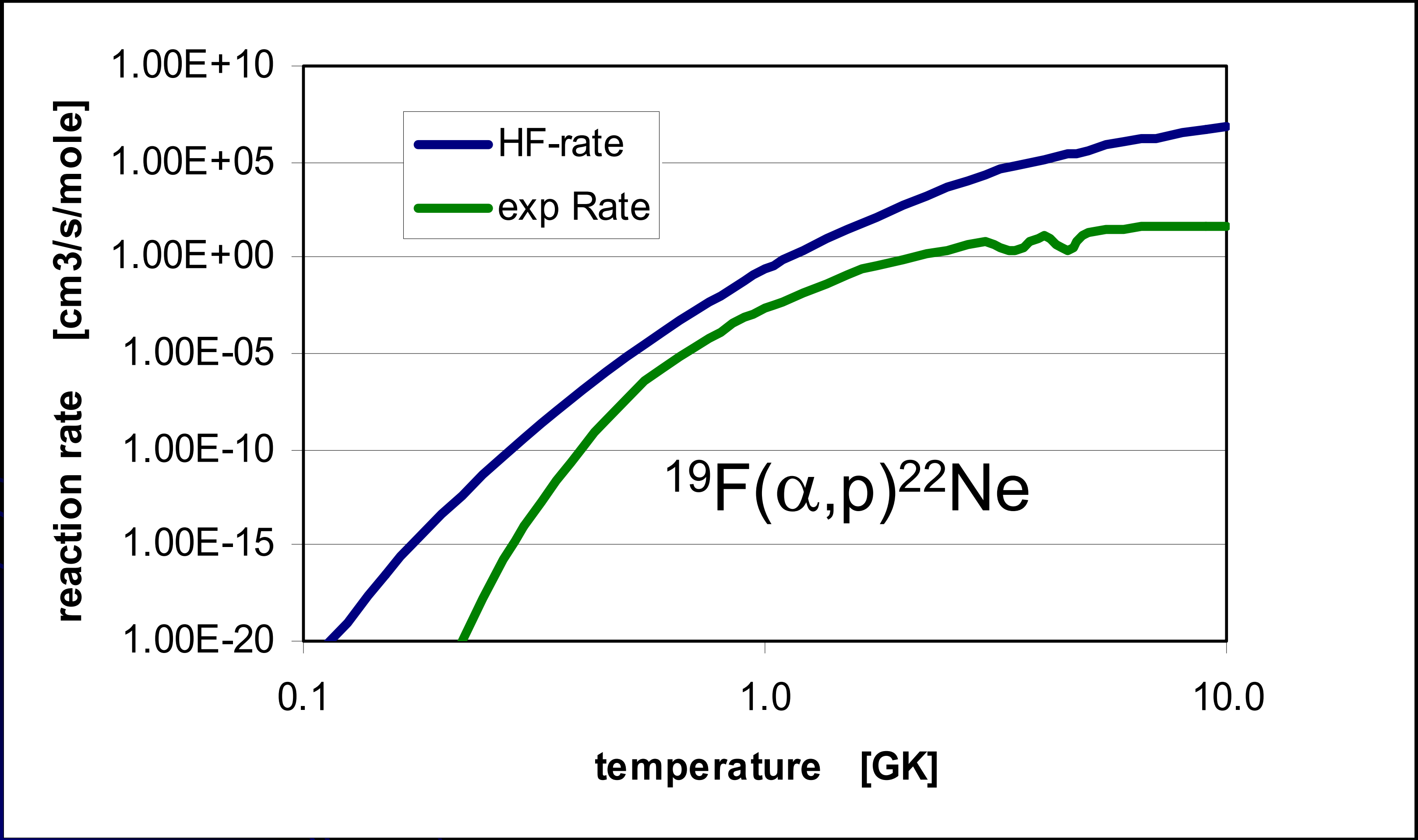
$$N_A < \sigma v > \propto T^{-3/2} \omega\gamma \exp (- E_R / kT)$$

σ: cross section

$\omega\gamma$: res. strength

E_R: res. energy

Typical temperature dependence of reaction rate of $^{19}F(\alpha,p)^{22}Ne$



exponential increase with temperature

Stellar reaction rates

$$r = \frac{1}{1+\delta_{pT}} Y_T Y_p \rho^2 \mathrm{N}_\mathrm{A}^2 < \sigma v >$$

reactions per s and cm^3

$$\lambda = \frac{1}{1+\delta_{pT}} Y_p \rho \mathrm{N}_\mathrm{A} < \sigma v >$$

reactions per s & Target nucleus

this is usually referred to
as the **stellar reaction rate**

units of stellar reaction rate $N_A<\sigma v>$: usually $cm^3/s/mole$

$$n_T = \rho \cdot N_A \cdot \frac{X_T}{A_T} = \rho \cdot N_A \cdot Y_T$$

X_T; mass fraction
Y_T: abundance

Change in Abundance

$$A + a \Rightarrow B$$

A reaction is a random process with a reaction probability (reaction rate) and follows the laws of radioactive decay:

Depletion of A

$$\frac{dn_A}{dt} = -n_A \lambda = -n_A Y_a \rho \, \mathrm{N_A} < \sigma v >$$

Formation of B

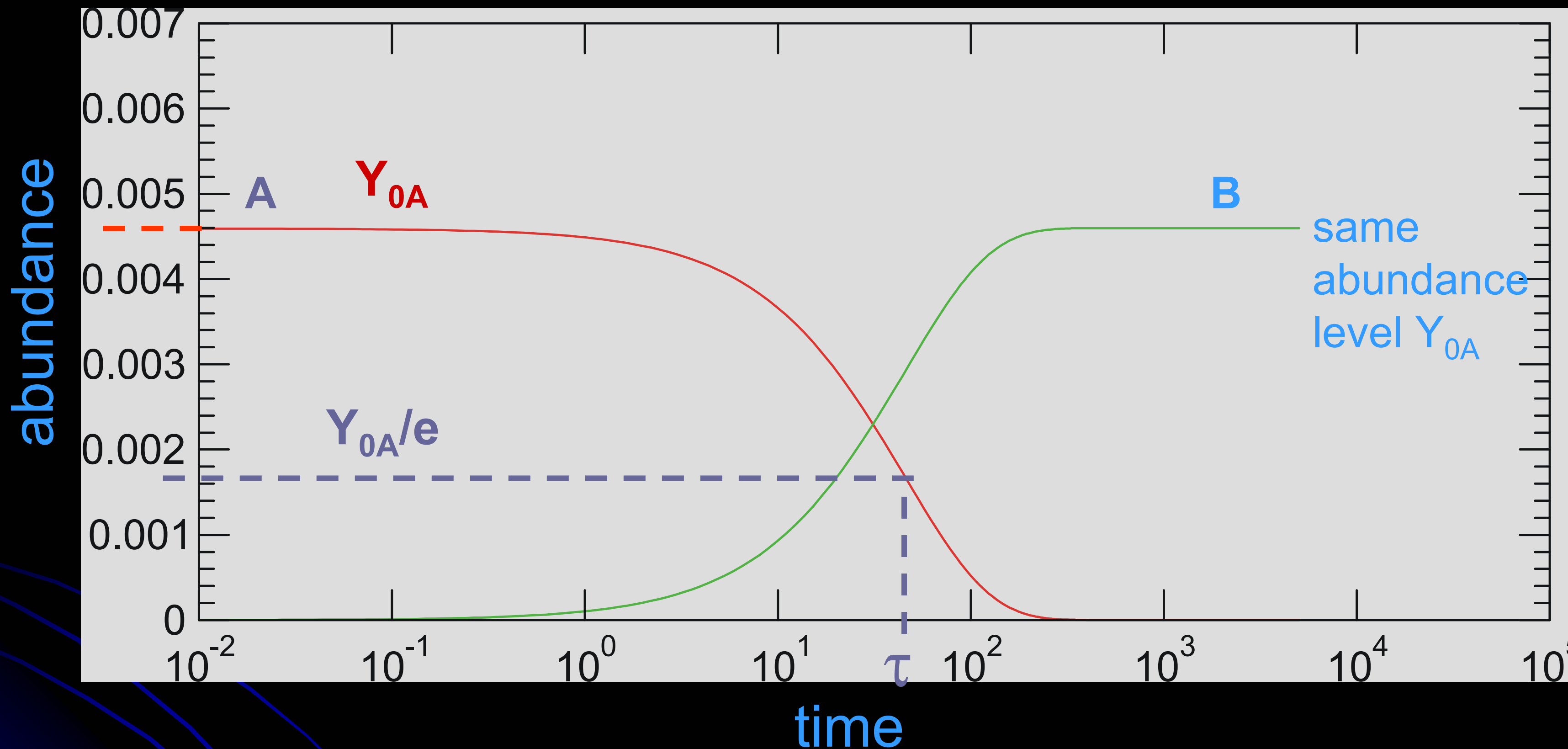
$$\frac{dn_B}{dt} = +n_A \lambda$$

consequently:

$$n_A(t) = n_{0A} \, \mathrm{e}^{-\lambda \mathrm{t}}$$

$$n_B(t) = n_{0A} (1 - \mathrm{e}^{-\lambda \mathrm{t}})$$

Stellar lifetime of nuclei



$$Y_A(t) = Y_{0A}\, e^{-\lambda t}$$

$$Y_B(t) = Y_{0A}(1 - e^{-\lambda t})$$

$$\tau = \frac{1}{\lambda} = \frac{1}{Y_a \rho N_A < \sigma v >}$$

Energy production

Reaction Q-value: Energy generated (if >0) by a single reaction

$$Q = c^2 \left(\sum_{\text{initial nuclei i}} m_i - \sum_{\text{final nuclei j}} m_j \right)$$

Difference between masses in entrance and exit channel

Energy generation: Energy generated per g and sec by a reaction:

$$\varepsilon = \frac{r \cdot Q}{\rho} = Q \cdot \frac{1}{1+\delta_{aA}} Y_A \cdot Y_a \cdot \rho \cdot \mathrm{N}_\mathrm{A}^2 < \sigma v >$$

Reaction Flow

Reaction flow is defined as the net # of nuclei converted in time T from species A to B via a specific reaction

$$F = \int_0^T \left(\frac{dY_A}{dt} \right)_{\text{via specific reaction}} dt = \int_0^T \lambda(t) Y_A(t) dt$$

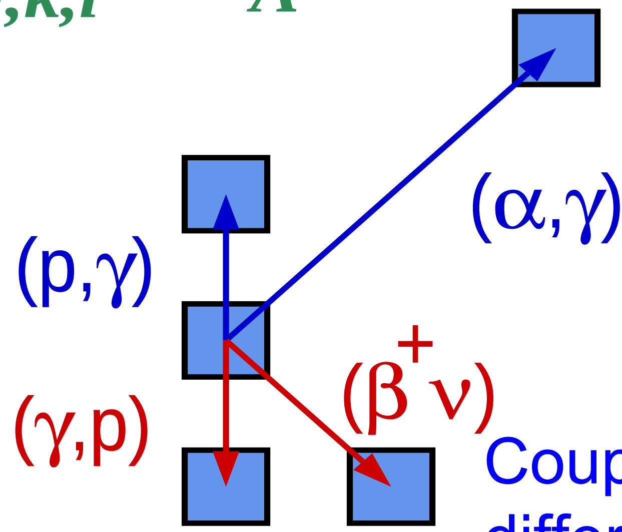
Reaction path is usually defined as the sequence of reactions with maximum reaction flow in a certain stellar environment!

Reaction Network Simulations

Change of istopic abundances:

$$\frac{dY_i}{dt} = \sum_j N_j^i \lambda_j Y_j + \sum_{j,k} N_{j,k}^i \rho N_A < j,k > Y_j Y_k +$$

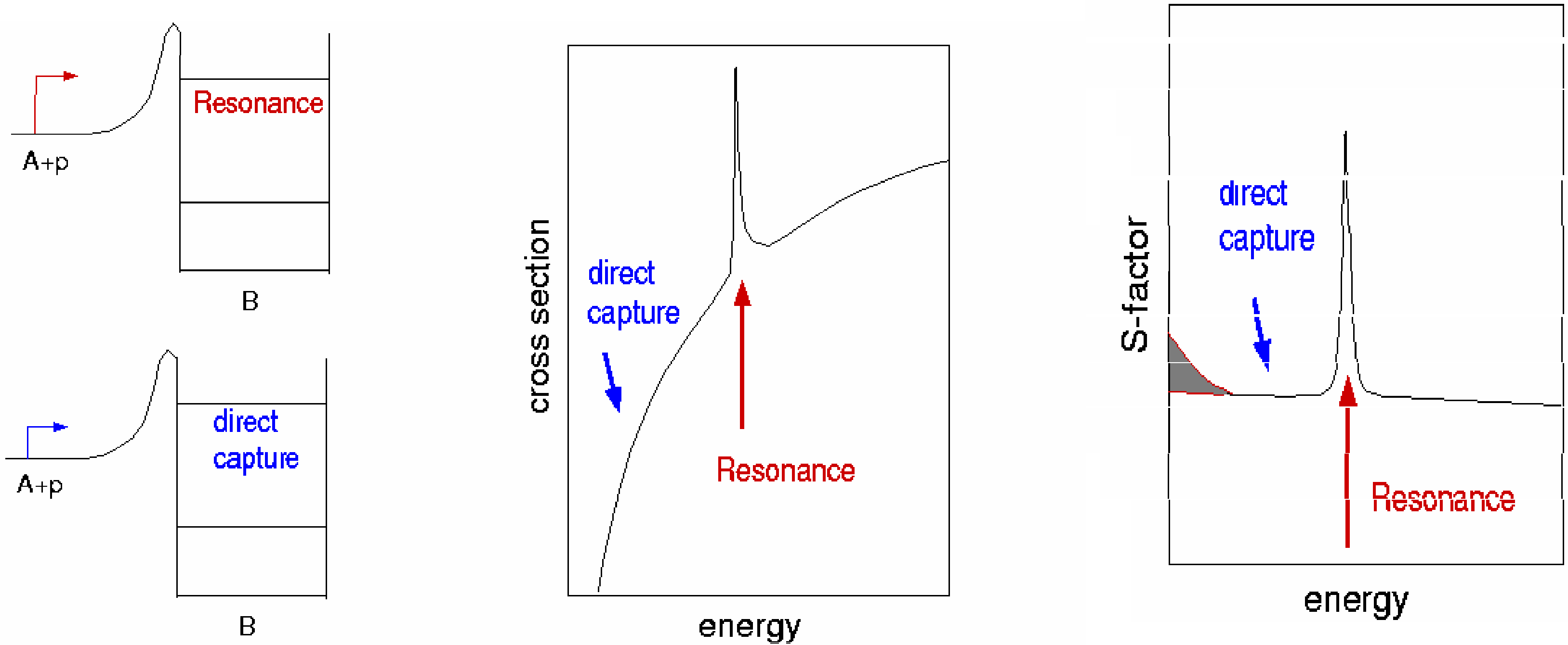
$$+ \sum_{j,k,l} N_{j,k,l}^i \rho^2 N_A^2 < j,k,l > Y_j Y_k Y_l$$



Coupled system of nonlinear differential equations

Reaction flux = net number of reactions:

Cross-Section and S-Factor

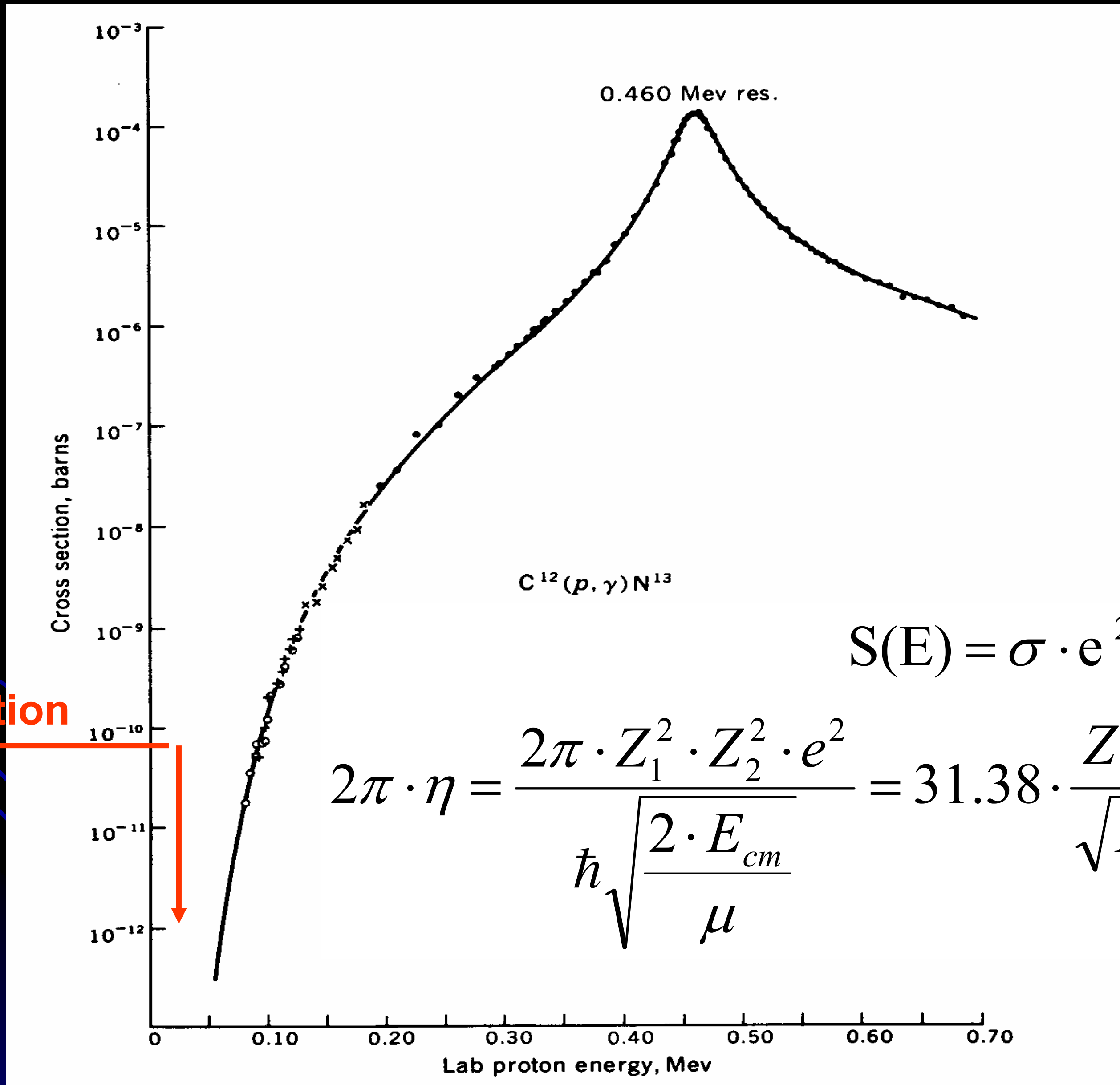


$$P_\ell = \left| \frac{\varphi_\ell(r=\infty)}{\varphi_\ell(r=R_0)} \right|^2 \Rightarrow P_\ell(E,R_0) = \frac{1}{F_\ell^2(E,R_0)+G_\ell^2(E,R_0)}$$

$$\ell = 0,\ E << E_C \Rightarrow P_0(E) \approx e^{-2\pi\eta}$$

S-factor to correct for Coulomb barrier: $S(E) = \sigma(E)\ E\ e^{2\pi\eta}$

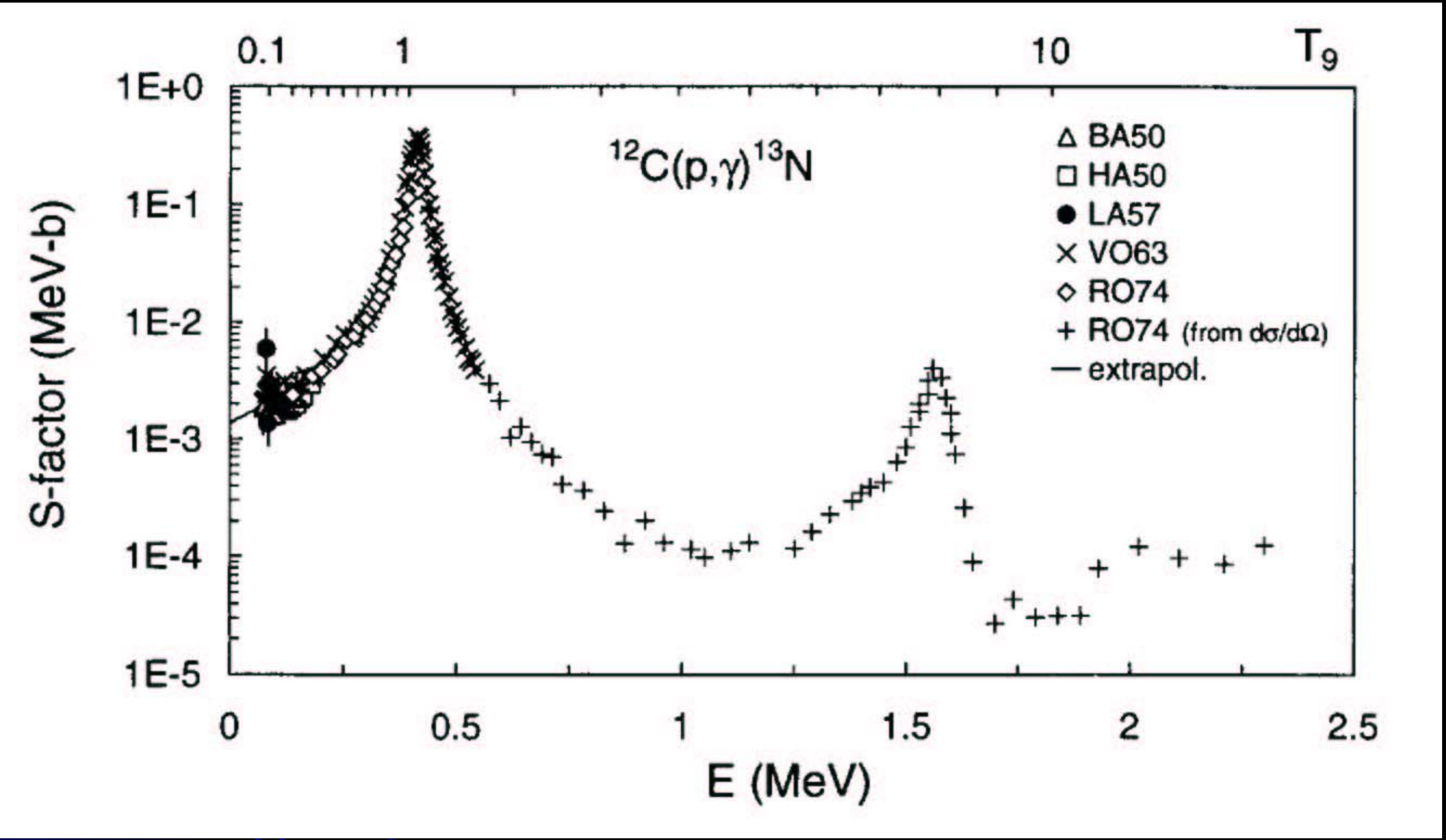
Example: $^{12}C(p,\gamma)$ cross section



$$S(E) = \sigma \cdot e^{2\pi \cdot \eta}$$

$$2\pi \cdot \eta = \frac{2\pi \cdot Z_1^2 \cdot Z_2^2 \cdot e^2}{\hbar \sqrt{\frac{2 \cdot E_{cm}}{\mu}}} = 31.38 \cdot \frac{Z_1^2 \cdot Z_2^2 \cdot \sqrt{\mu}}{\sqrt{E_{cm}\,[MeV]}}$$

S-factor Conversion



From the **NACRE compilation** of charged particle induced reaction rates on stable nuclei from H to Si (Angulo et al. Nucl. Phys. A 656 (1999) 3

Reaction rate from S-factor

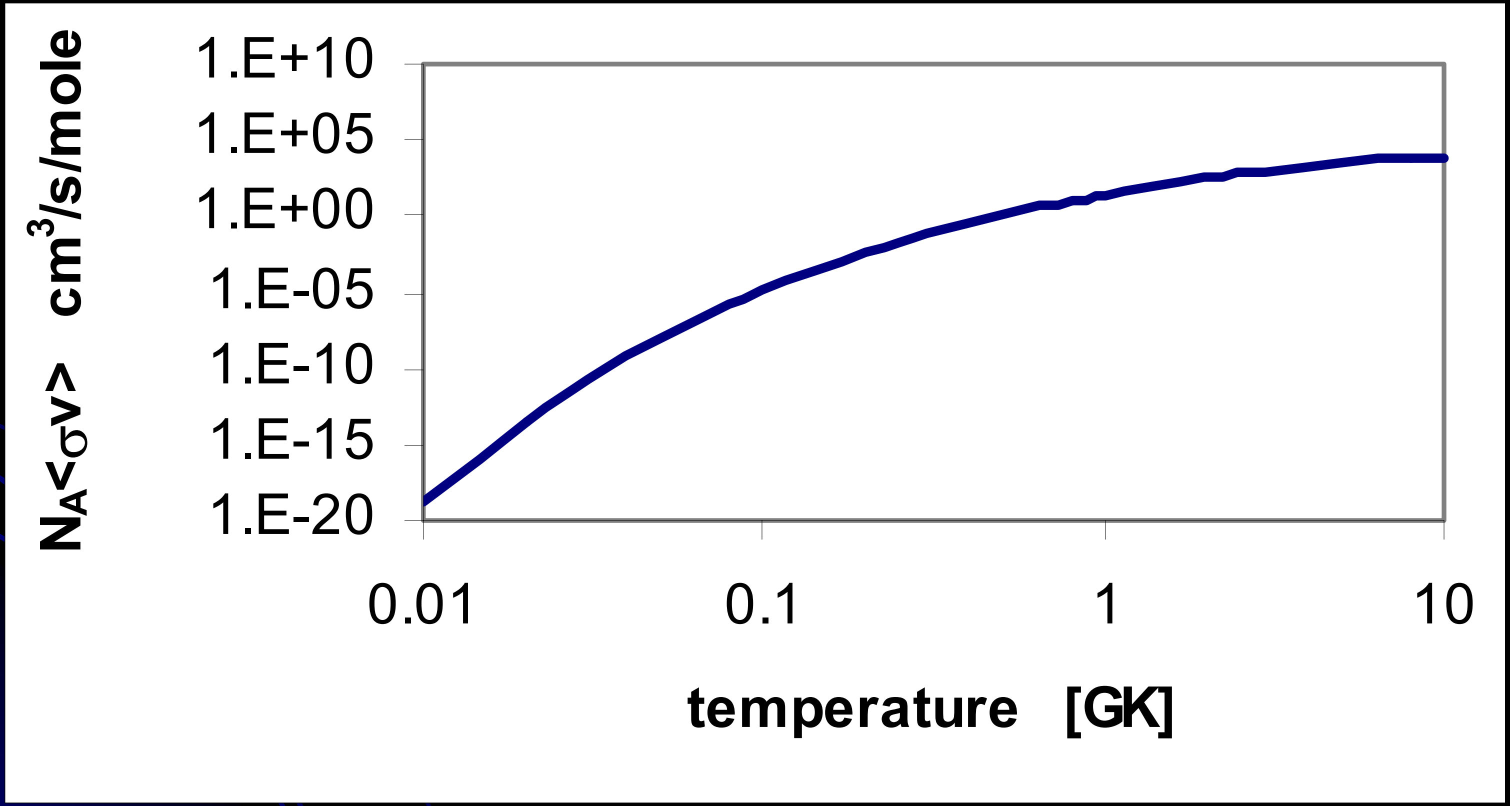
If S-factor ~ constant over the Gamow range the rate is calculated in terms of the S-factor

$S(E)=S(E_0)$

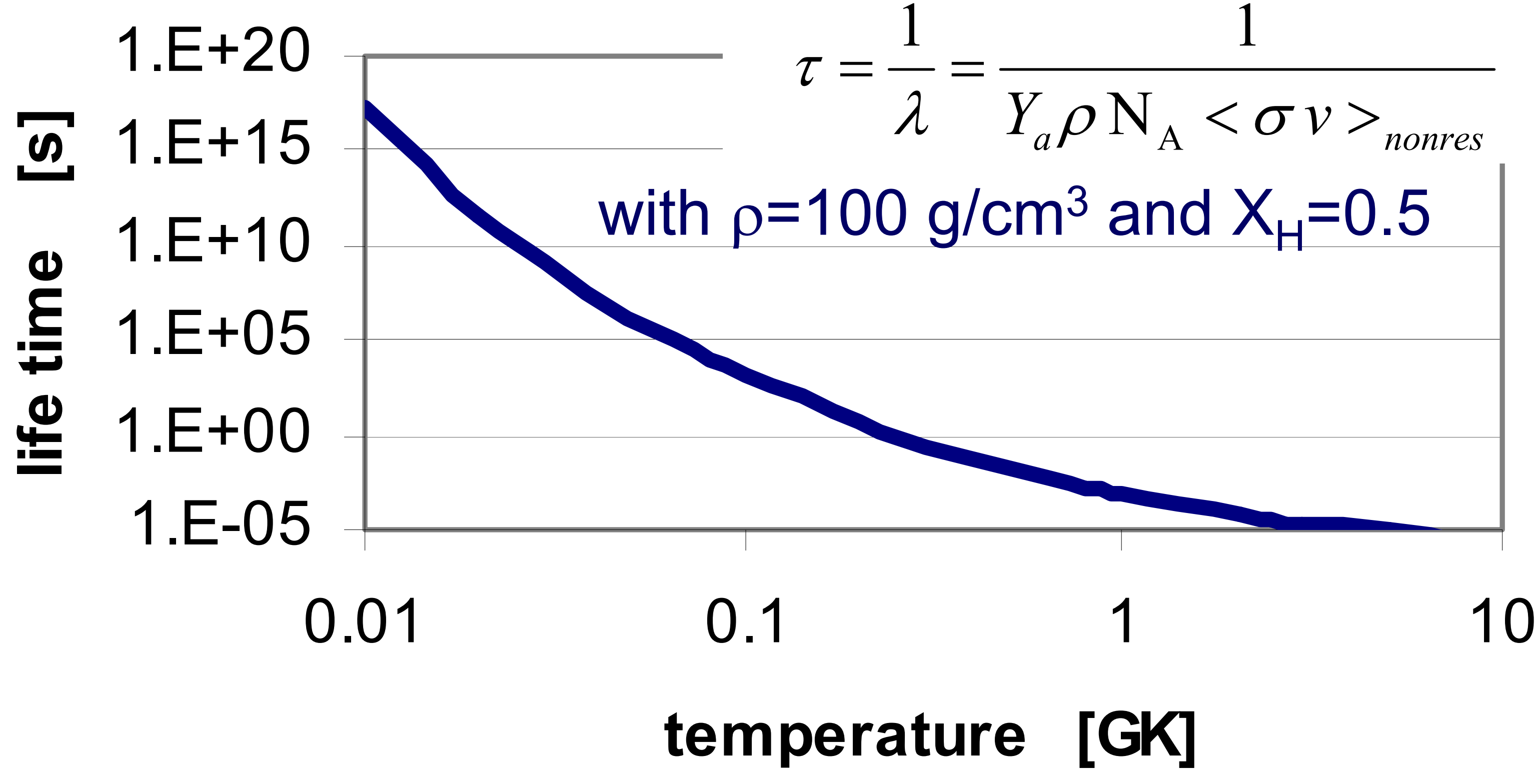
$$N_A < \sigma v >= 7.83 \cdot 10^9 \left(\frac{Z_1 Z_2}{\mu T_9^2} \right)^{1/3} S(E_0)[\text{MeV barn}]\ e^{-4.2487 \left(\frac{Z_1^2 Z_2^2 \mu}{T_9} \right)^{1/3}}$$

Example: $^{12}C(p,\gamma)^{13}N$

Calculate the reaction rate as function of temperature!



Calculate life time of ^{12}C



$$\tau = \frac{1}{\lambda} = \frac{1}{Y_a \rho N_A < \sigma v >_{nonres}}$$

Determine ^{12}C life time in the sun T=0.015 GK!

$\tau \approx 6 \cdot 10^{13}$ s $\approx 2 \cdot 10^{6}$ y …

Resonant Reaction Rate

$$N_A\langle\sigma v\rangle = 1.54\cdot 10^{11}\cdot\omega\gamma\,[MeV]\cdot\left(\frac{1}{\mu\cdot T_9}\right)^{3/2}\cdot e^{-\left(\frac{11.605\cdot E_R[MeV]}{T_9}\right)}$$

$$\omega\gamma = \frac{2(J+1)}{2(j_p+1)\cdot 2(j_T+1)}\cdot\frac{\Gamma_{in}\cdot\Gamma_{out}}{\Gamma_{tot}} \qquad \Gamma_{tot} = \sum_i \Gamma_i$$

at low energy astrophysical conditions: $\Gamma_{in} << \Gamma_{out} \approx \Gamma_{tot}$

$$\Gamma_{in} = \Gamma_{p,\alpha} = T_{V_c,V_\ell}\cdot\Theta_{p,\alpha}$$

$$\Theta_{p,\alpha} << 1$$

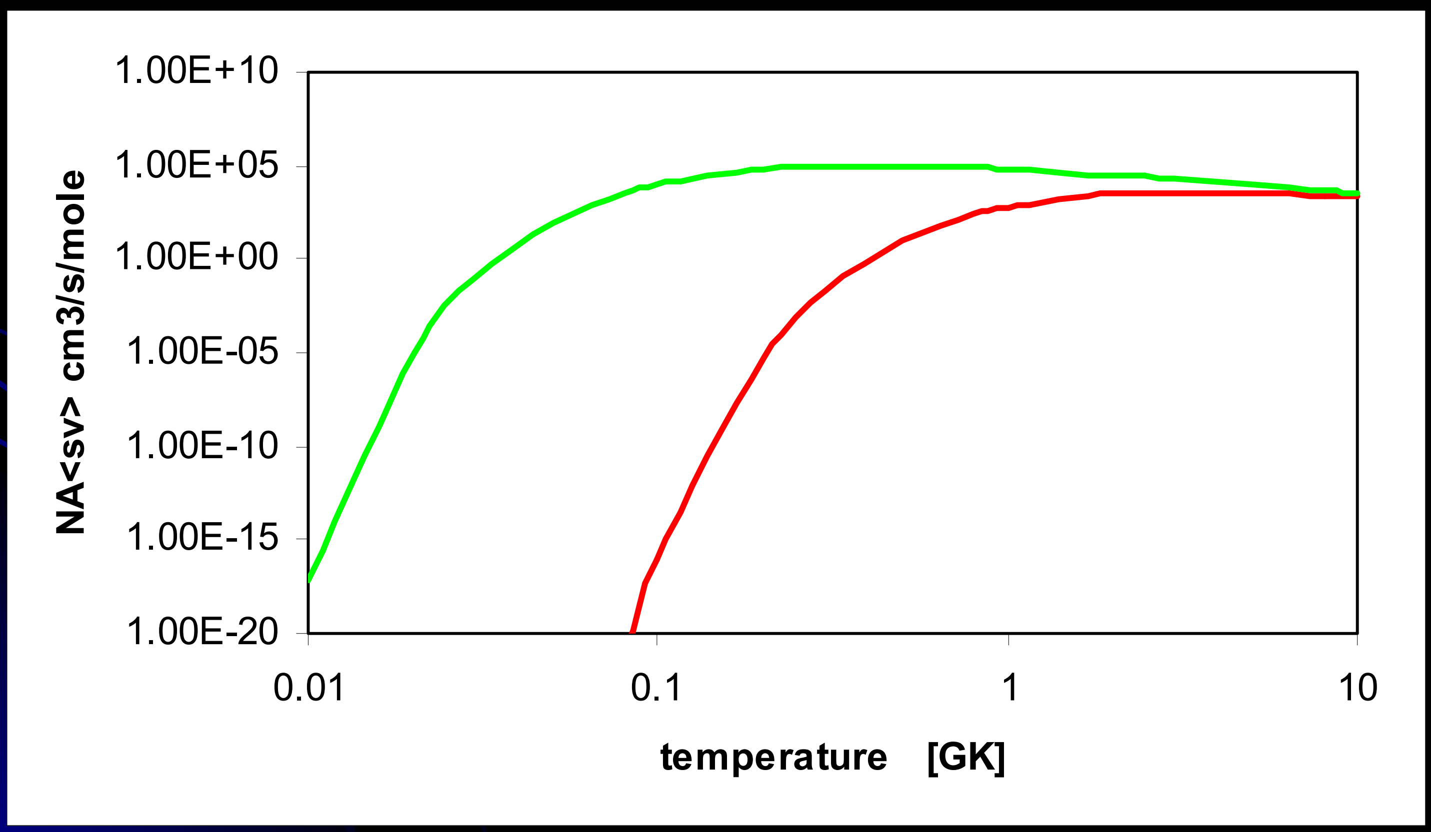
spectroscopic factor, depends on resonant state configuration

Calculate $^{12}C(p,\gamma)^{13}N$ resonance

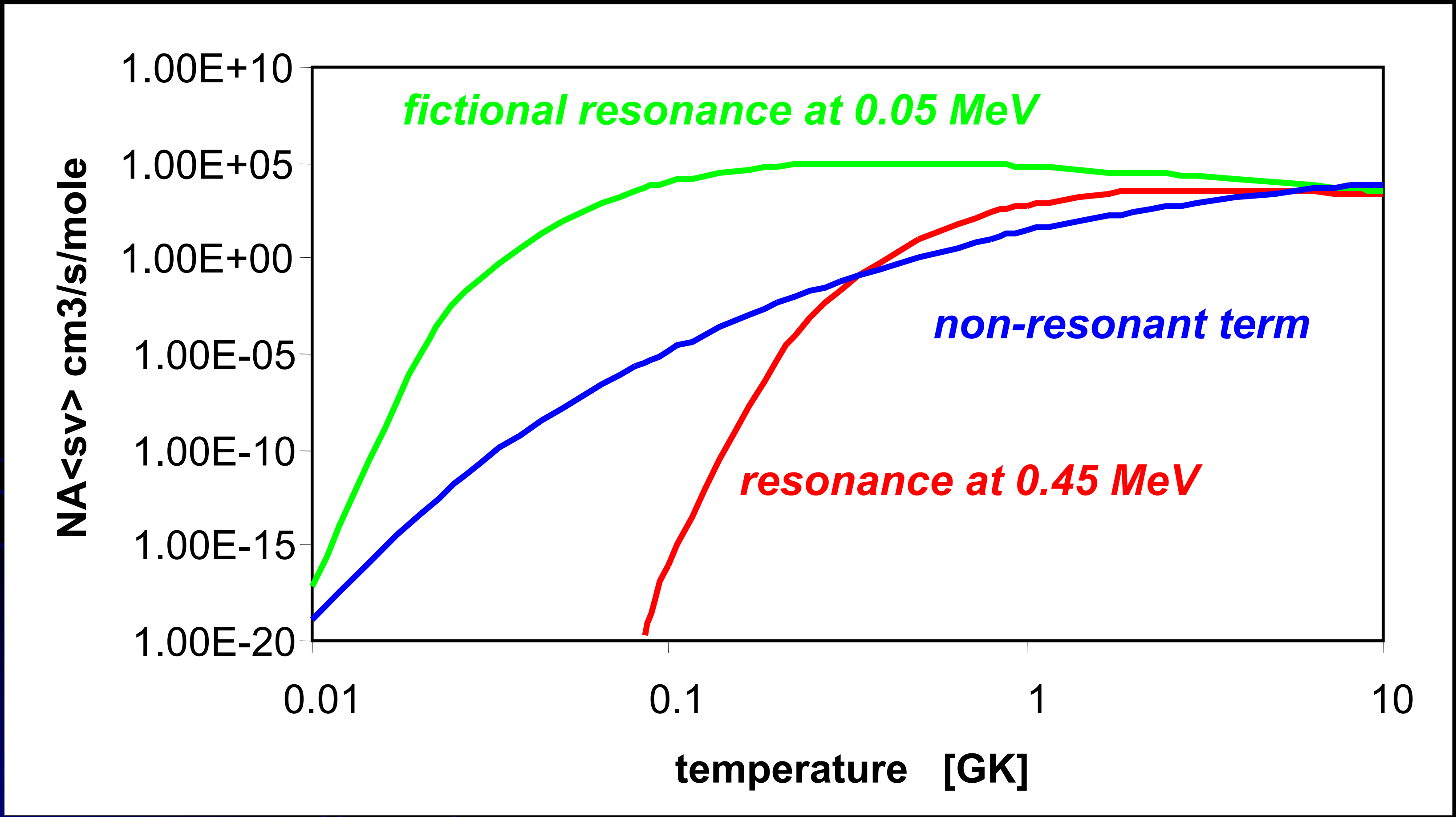
$\omega\gamma$ = 0.65 eV= 6.5 10^{-7} MeV

experimental value: E_r^{cm}= 0.45 MeV

assumed value: E_r^{cm}= 0.05 MeV

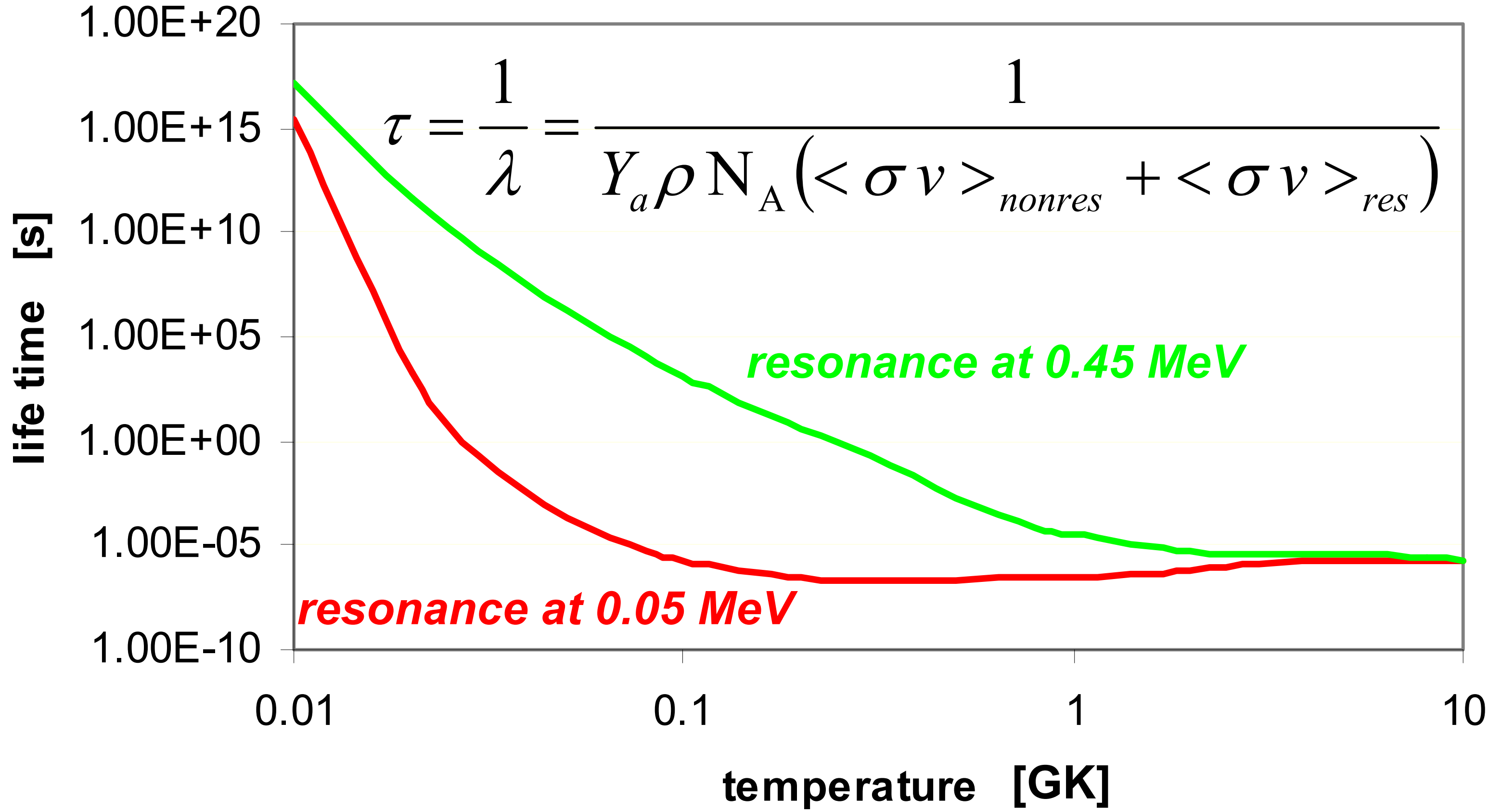


Comparison between resonant and non-resonant contributions



strong dependence in resonance energy!!!

Impact on life time of ^{12}C



$$\tau = \frac{1}{\lambda} = \frac{1}{Y_a \rho N_A \left(<\sigma v>_{nonres} + <\sigma v>_{res} \right)}$$

with fictional resonance at 0.05 MeV life time of solar ^{12}C would drop to ≈ 1 s only!

Reaction Rates for inverse Processes

$$\frac{\sigma_{Aa}}{\sigma_{Bb}} = \frac{(2j_B+1)(2j_b+1)}{(2j_A+1)(2j_a+1)} \cdot \frac{m_B \cdot m_b \cdot E_{Bb}}{m_A \cdot m_a \cdot E_{Ab}}$$

detailed balance

The ratio of reaction rate to inverse reaction rate depends on Q and T!

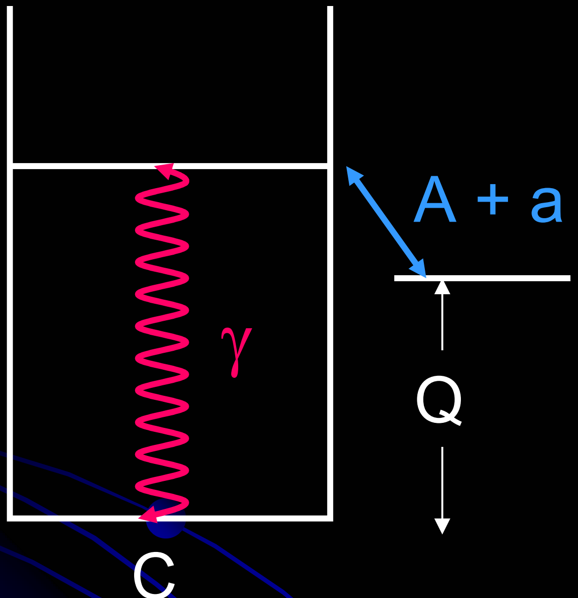
$$\langle \sigma \upsilon \rangle_{Aa} = \sqrt{\frac{8\pi}{\mu_{Aa}}} \cdot (kT)^{-3/2} \cdot \int_0^\infty \sigma_{Aa} \cdot E_{Aa} \cdot e^{\left(-\frac{E_{Aa}}{kt}\right)} dE_{Aa}$$

$$\langle \sigma \upsilon \rangle_{Bb} = \sqrt{\frac{8\pi}{\mu_{Bb}}} \cdot (kT)^{-3/2} \cdot \int_0^\infty \sigma_{Bb} \cdot E_{Aa} \cdot e^{\left(-\frac{EBb}{kt}\right)} dE_{Bb}$$

$$E_{Bb} = Q + E_{Aa}$$

$$\langle \sigma \upsilon \rangle_{Bb} = \left(\frac{(2j_A+1)(2j_a+1)}{(2j_B+1)(2j_b+1)}\right) \cdot \left(\frac{\mu_{Aa}}{\mu_{Bb}}\right)^{3/2} \cdot e^{\left(-\frac{Q}{kT}\right)} \cdot \langle \sigma \upsilon \rangle_{Aa}$$

Photodisintegration Processes



$$A + a \Rightarrow C + \gamma$$

$$\gamma + C \Rightarrow A + a$$

equilibrium conditions between forward and reverse reaction!

$$N_A N_a \cdot \langle \sigma \upsilon \rangle_{Aa} = N_C \cdot \lambda_\gamma$$

Saha Equation

$$\left(\frac{N_A N_a}{N_C} \right)_{eq} = \frac{(2\pi \cdot \mu_{Aa} \cdot kT)^{3/2}}{h^3} \cdot \left(\frac{(2j_A + 1)(2j_a + 1)}{(2j_C + 1)} \right) \cdot e^{\left(-\frac{Q}{kT} \right)}$$

photo disintegration rate in units [1/s]

$$\lambda_\gamma = \frac{1}{\tau_\gamma} = \frac{(2\pi \cdot \mu_{Aa} \cdot kT)^{3/2}}{h^3} \cdot \left(\frac{(2j_A + 1)(2j_a + 1)}{(2j_C + 1)} \right) \cdot e^{\left(-\frac{Q}{kT} \right)} \cdot \langle \sigma \upsilon \rangle_{Aa}$$

Conclusion

- stellar reaction rates depend on low energy cross sections
- reaction rates determine
  - stellar nucleosynthesis
  - timescale of nuclear burning processes
  - stellar energy production
- reaction rates depend on cross section and resonance parameters in Gamow range