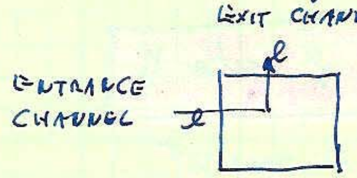
MULTI-CHANNEL R-MATRIX THEORY

SECTION V LT. p. 282.

Generalize $\boxed{R = \sum_\lambda{}' \frac{\gamma_\lambda^2}{E_\lambda - E}}$ Single Channel R-Function.

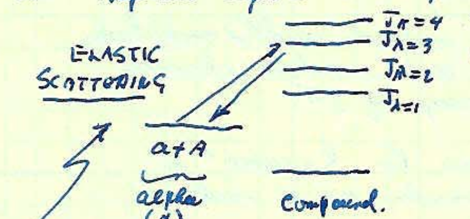
It is instructive to regard this eqn. as a 1×1 matrix.



where ℓ is the channel index

We assume we enter from the Left and exit from the top

The "Physical System" we represent is



Group of Same J^π ***

Since we have a spin-zero system then the single channel is defined by the orbital ang. mom. ℓ.

Channel Spin = $I_1 + I_2 = 0$

Thus our 1×1 matrix can be written as

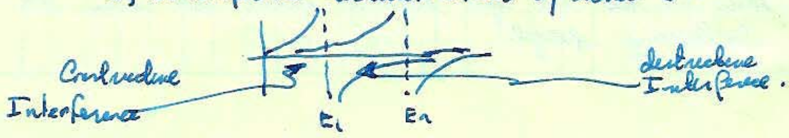
(EXIT CHANNEL)

$$\left(\begin{matrix}\text{Entrance}\\ \text{Channel}\end{matrix}\right) \begin{array}{c|c} & \ell \\ \hline \ell & \boxed{\sum_\lambda \frac{\gamma_\lambda^2}{E_\lambda - E}} \end{array} = \begin{array}{c|c} & \ell \\ \hline \ell & \boxed{\frac{\gamma_1^2}{E_1 - E} + \frac{\gamma_2^2}{E_2 - E} + \dots + \frac{\gamma_n^2}{E_n - E}} \end{array}$$

Things to Note:

1) As we saw previously, we have defined a finite BC. Bc in our discussions of the R-function, so remember the Bc is defined for a given (i.e. first level)

2) Interference between states of same J^π.



THE MATRIX DEFN. OF THE R-MATRIX

$$\boxed{R = \sum_{\lambda} (\gamma_\lambda \times \gamma_\lambda)/(E_\lambda - E)}$$

where $\boxed{(\gamma_\lambda \times \gamma_\lambda) \equiv \text{Matrix Square of vector } \gamma_\lambda}$

Then for the GROUP OF LEVELS with the same J^π

$$\gamma_\lambda = (\gamma_{\lambda c_1}\ \gamma_{\lambda c_2}\ \gamma_{\lambda c_3} \cdots \gamma_{\lambda c_1'}\ \gamma_{\lambda c_2'} \cdots)$$

The components are ALL the reduced width amplitudes for all the possible s-l and reaction channels associated with the group of J^π.

The matrix square is defined as the "square" of the vector γ_λ with the latter consider as a matrix.

$$\therefore \text{ Matrix Square} \equiv \begin{pmatrix} \gamma_{\lambda c_1} \\ \gamma_{\lambda c_2} \\ \vdots \\ \gamma_{\lambda c_i'} \end{pmatrix} (\gamma_{\lambda c_1}\ \gamma_{\lambda c_2} \cdots \gamma_{\lambda c_i'})$$

If $A \times B$ exists

A & B must be (conformable)

number of elements in ROW A = number of elements in COL B

$$\equiv \begin{pmatrix} \gamma_{\lambda c_1}\gamma_{\lambda c_1} & \gamma_{\lambda c_1}\gamma_{\lambda c_2} & \cdots \\ \gamma_{\lambda c_2}\gamma_{\lambda c_1} & \gamma_{\lambda c_2}\gamma_{\lambda c_2} & \cdots \\ & & \end{pmatrix}$$

Thus we see we get exactly what we have on following page.

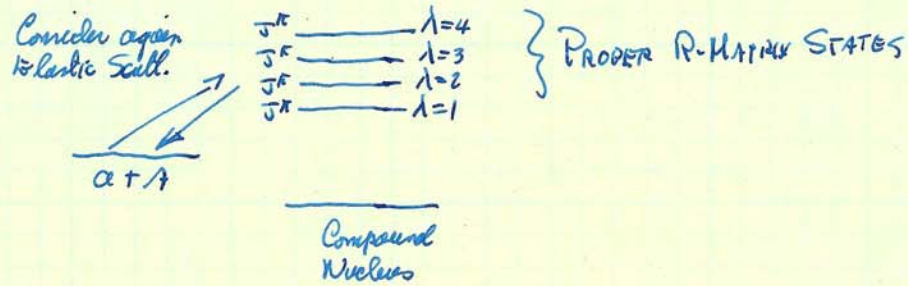
We specify the General Boundary Condition

$$\frac{D_{\lambda c}}{V_{\lambda c}} \left(\equiv \frac{\delta_{\lambda c}}{\gamma_{\lambda c}} \right) = B_c$$ which as shown, leads to the definition of the multi-channel R-matrix.

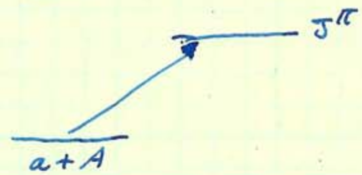
$$\boxed{R_{c'c} = \sum_{\lambda}' \frac{\gamma_{\lambda c'} \gamma_{\lambda c}}{E_\lambda - E}}$$ For a given J^π

Let us now explore this in terms of actual channels we might get in a nuclear reaction.

NOTES:- 1) Let us look at an energy level diagram



2) The "Channels" The s-l channels.



The channel spin s combines with the allowed l-value to form the state J^π

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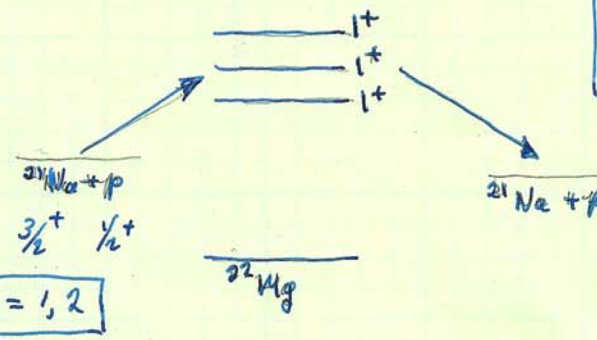
eg. $^{21}Na(p,p)^{21}Na$



$\therefore \boxed{s = 1, 2}$

FOR

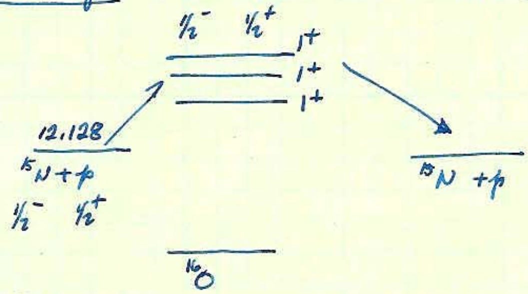
$s=1$	$l=0$
	$l=2$
$s=2$	$l=2$

$\therefore$ THERE ARE 3 - ENTRANCE CHANNELS AND 3 - EXIT CHANNELS.

ELASTIC SCATTERING, THE s-ℓ CHANNELS

Consider Simple Case of $^{15}N + p$. i.e. $^{15}N(p,p)^{15}N$

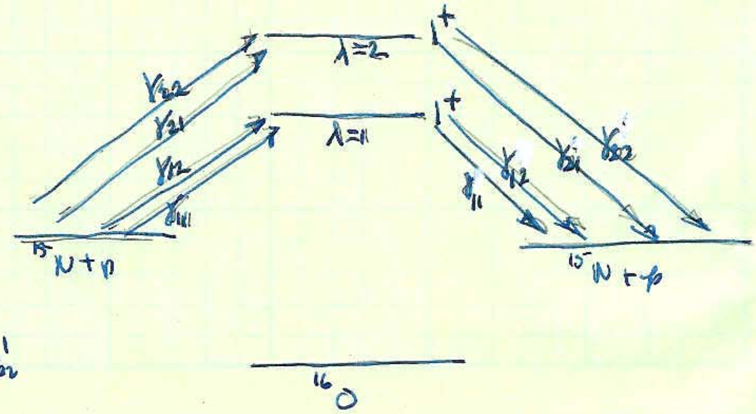
$\therefore\ s = (0,1)^-$



$$\left.\begin{array}{l} \text{for } s=0,\ \ell=1 \\ \quad\ \ s=1,\ \ell=1 \end{array}\right\} \text{Two channels allowed.}$$

$\therefore$ ELASTIC SCATTERING WILL BE A 2×2 MATRIX.

			ENTRANCE CHANNELS	EXIT CHANNELS: $(\gamma_{21}$ / $(\gamma_{11}$; $s=0$, $\ell=1$	$\gamma_{22})$ / $\gamma_{12})$; $s=1$, $\ell=1$
$\vec{\gamma}_2$	$\vec{\gamma}_1$				
γ_{21}	γ_{11}	$s=0$, $\ell=1$		$\dfrac{\gamma_{11}\gamma_{11}}{E_1-E} + \dfrac{\gamma_{21}\gamma_{21}}{E_2-E} + \cdots$	$\dfrac{\gamma_{11}\gamma_{12}'}{E_1-E} + \dfrac{\gamma_{21}\gamma_{22}'}{E_2-E} + \cdots$
γ_{22}	γ_{12}	$s=1$, $\ell=1$		$\dfrac{\gamma_{12}\gamma_{11}}{E_1-E} + \dfrac{\gamma_{22}\gamma_{21}'}{E_1-E} + \cdots$	$\dfrac{\gamma_{12}\gamma_{12}'}{E_1-E} + \dfrac{\gamma_{22}\gamma_{22}}{E_2-E} + \cdots$



FOR ELASTIC

$\gamma_{11} = \gamma_{11}'$ $\quad \gamma_{22} = \gamma_{22}'$

$\gamma_{12} = \gamma_{12}'$

$\gamma_{21} = \gamma_{21}'$

NOTE: One of the reasons it makes sense to ~~group~~ arrange the compound (proper) states ~~of the~~ in groups of the same J^π, is that

ALL LEVELS OF THE SAME J^π HAVE THE SAME ~~S~~-l CHANNELS

NOTE: The sums $\sum_\lambda' \frac{\gamma_{\lambda c}\gamma_{\lambda c'}}{E_\lambda - E}$ are, computationally easy to do.

$\therefore$ If there are FEW CHANNELS
MANY LEVELS FOR SAME J^π
USE R-MATRIX

The above statement may seem bizarre, but there is a variant of the R-Matrix, called the LEVEL (or A) Matrix which is used more often than the R-Matrix — see next lecture

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FOR CALCULATIONAL PURPOSES

Calculate the Elements of the R-Matrix.

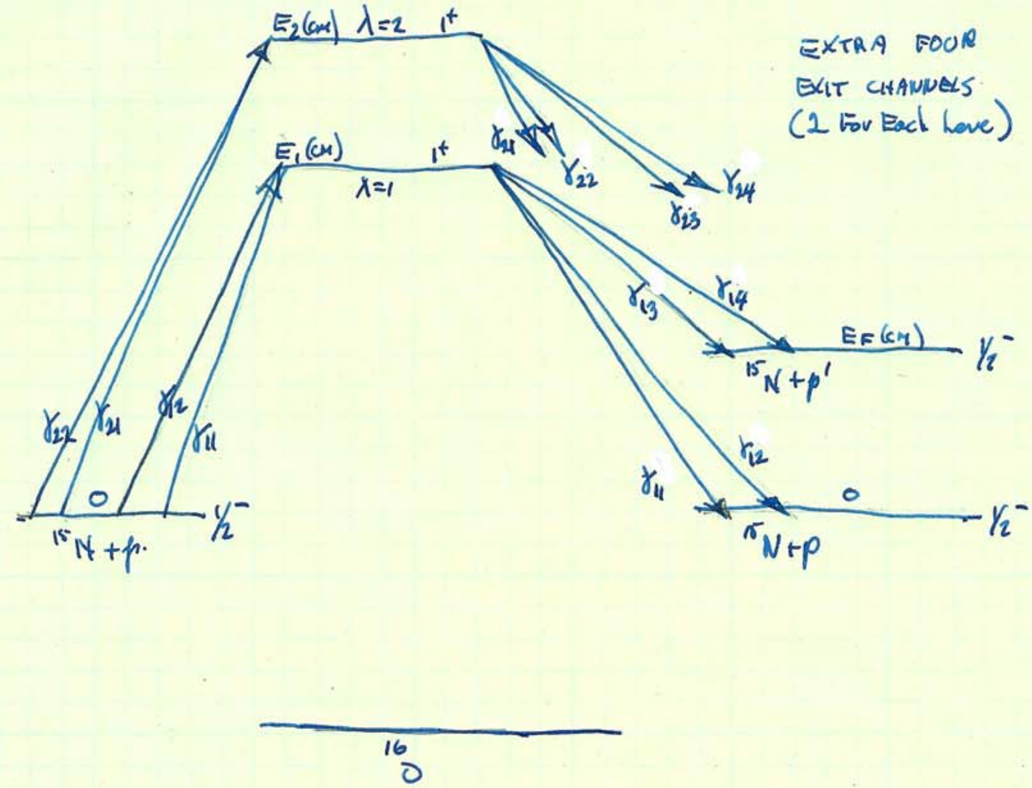
(i) calculate all allowed combinations of s-l
(ii) calculate sums for each element.
} this is relatively easy.

(iii) keep track of all s- and l-values.

ADD REACTION CHANNELS TO BASIC ELASTIC SCATT. R-MATRIX

We will assume the same example as above $^{15}N+p$
We will add Inelastic Scattering to a State in ^{15}N of $J^\pi = 1/2^-$ (the same as the J^π of the ground state — then we do not have to calculate the allowed values of s, l for the new exit channels.

Considering the energy level scheme again, we have the following



THE R-MATRIX FOR THE ABOVE EXAMPLE

Now that we have TWO MORE CHANNELS (For Each Level) WE WILL Have a 4×4 matrix where each element is a sum over levels.

NOW ADD MORE EXIT REACTION CHANNELS

		ELASTIC $\gamma_{\lambda1}$ $s=0$ $l=1$	$\gamma_{\lambda2}$ $s=1$ $l=1$	INELASTIC $\gamma_{\lambda3}$ $s=0$ $l=1$	$\gamma_{\lambda4}$ $s=$ $l=1$
ELASTIC	$\gamma_{\lambda1}$ $s=0$ $l=1$	$\sum_\lambda \frac{\gamma_{\lambda1}\gamma_{\lambda1}}{E_\lambda-E}$	$\sum_\lambda \frac{\gamma_{\lambda1}\gamma_{\lambda2}}{E_\lambda-E}$	$\sum_\lambda \frac{\gamma_{\lambda1}\gamma_{\lambda3}}{E_\lambda-E}$	$\sum_\lambda \frac{\gamma_{\lambda1}\gamma_{\lambda4}}{E_\lambda-E}$
	$\gamma_{\lambda2}$ $s=1$ $l=1$	$\sum_\lambda \frac{\gamma_{\lambda2}\gamma_{\lambda1}}{E_\lambda-E}$	$\sum_\lambda \frac{\gamma_{\lambda2}\gamma_{\lambda2}}{E_\lambda-E}$	$\sum_\lambda \frac{\gamma_{\lambda2}\gamma'_{\lambda3}}{E_\lambda-E}$	$\sum_\lambda \frac{\gamma_{\lambda2}\gamma'_{\lambda4}}{E_\lambda-E}$
INELASTIC	$\gamma_{\lambda3}$	$\sum_\lambda \frac{\gamma_{\lambda3}\gamma_{\lambda1}}{E_\lambda-E}$	$\sum'_\lambda \frac{\gamma_{\lambda3}\gamma_{\lambda2}}{E_\lambda-E}$	$\sum'_\lambda \frac{\gamma_{\lambda3}\gamma_{\lambda3}}{E_\lambda-E}$	$\sum_\lambda \frac{\gamma_{\lambda3}\gamma_{\lambda4}}{E_\lambda-E}$
	$\gamma_{\lambda4}$	ETC	ETC	ETC	ETC.

PHYSICAL CHANNELS

INELASTICS IN - ELASTIC OUT

INELASTIC IN INELASTIC OUT

OTHER EXIT REACTION CHANNELS

EXERCISE FOR STUDENT

e.g. Assume you can add ~~4~~ reaction exit channels for, say neutrons and for deuterons. Sketch the form of the R-Matrix.

OTHER GROUPS OF J^π

EXERCISE FOR STUDENT

What do we do about OTHER groups of J^π ???

FOR BREIT-WIGNER AFICIONADOS

EACH reaction pathway requires ONE B-W curve !!

REACTION PATHWAYS — Previous Example.

Consider physical channels only:
For EACH level, there are two possible entrance channels and FOUR exit channels..
THUS there is a possibility of $2 \times 4 = 8$ REACTION PATHWAYS

NOTES: (1) Each reaction pathway (channel in $\longrightarrow$ channel out) is a sum over levels of same J^{π}

(2) Each pathway is associated with the corresponding element of the collision-matrix U.
Thus, there is a correspondence between the elements of U and R.

(3) Each pathway, through the $\sum_{\lambda}$, includes interference effects between levels of same J^{π}

(4) Each element of the collision matrix U (and hence the elements of the R-matrix) contribute an AMPLITUDE to the final differential cross section σ.

(5) Each reaction pathway has a set of entrance channel s and l-values, and exit channel s' and l'-values.
So that a reaction channel is characterized by:- - -
$(s_c, l_c, s'_{c'}, l_{c'}) \Longrightarrow U_{cc'}$, and the final $d\sigma/d\Omega$

$$d\sigma/d\Omega \approx \left| \sum_{cc'}' U_{cc'} \right|^2$$ — THIS IS A HORRIBLE EQN TO CONTEMPLATE CALCULATING.

(6) CLEBSCH-GORDON & RACAH COEFFICIENTS

Expanding our sum

$$d\sigma/d\Omega \sim \left| U_{11} + U_{12} + U_{13} + U_{14} + U_{21} + U_{22} + U_{23} + U_{24} \right|^2$$

$U_{11}(s_1, l_1, s'_1, l'_1)$ $\qquad$ $U(s_2, l_2, s'_3, l'_3)$

IN THE SQUARE, ONE GETS CROSS PRODUCTS **
It is essential in Q.M., that if one has a product of two quantities which are characterized by the angular momentum quantum Nos. $\Rightarrow$ CONSERVATION OF ANGULAR MOMENTUM.

It is, of course, in the expressions involving addition of angular momenta that the C-G and RACAH coefficients emerge. Note particularly that each reaction path-way, and hence each U, involve an entrance and exit l-value. These products can be expanded in Legendre Polynomials of order L, so that the final angular part of the differential cross sections are expansions over Legendre Polynomial.

<u>WE WILL DO NONE OF THIS</u> !!!

For those who feel particularly masochistic, see AZURE, SECTION

<u>WHAT AZURE DOES (IN PART)</u>

Azure will calculate all possible s and l-values for all possible entrance ~~exit~~ reactions; ~~use~~ H (see AZURE SECTION), use these to calculate all ~~all~~ BC's matrix ~~and~~, L_0's matrix, ~~diff~~ ~~for~~ and necessary Coulomb functions; matrix and calculate all A-matrix and U-matrix elements. Azure uses only the information in the INPUT file which consists of J^π for the participating particles and levels, the values of reduced widths, the entrance & exit Q-values, the Z and M of the participating particles. The only requirement on the user is to get the 'input file' correct!

We will ~~does~~ discuss further AZURE CALCULATIONS IN LECTURE 3 & 4.

<u>Now on the the relationship between U and R</u>

Now Good Old Erich has done the work for us:-

$$U^J = \Omega W^J \Omega \qquad \text{eqn. 1.5 p. 289}$$

$$W^J = 1 + B^{1/2}(1 - R^J L_0)^{-1} R_J B^{1/2} w \qquad \text{eqn. 1.6a p. 289}$$

These are the only 2 eqns we actually use from this section

As a first example of the use of the these eqns, we will consider a hypothetical case of a reaction that has

1 - Level   <u>ONE LEVEL — TWO CHANNELS</u>

1 - s-l channel

2 - Reaction Exit channels, e.g. elastic & inelastic.   Prime ⇒ inelastic

Thus

USE SIMPLIFIED NOTATION FOR γ's.

The R-matrix will be a 2x2 Matrix

$$
\begin{array}{ccc}
 & \text{ELASTIC} & \text{INELASTIC} \\
 & \gamma & \gamma' \\
\text{ELASTIC} \ \ \gamma & \dfrac{\gamma\gamma}{E_1 - E} & \dfrac{\gamma\gamma'}{E_1 - E} \\
\text{INELASTIC} \ \ \gamma' & \dfrac{\gamma'\gamma}{E_1 - E} & \dfrac{\gamma'\gamma'}{E_1 - E}
\end{array}
$$

Physical Channels.

where ' indicates inelastic channel.

<u>We will Need the Diagonal Matrix L_o and $B^{1/2}$</u>

Remember L_o is calculated for <u>exit</u> channels.

$L_o = \left(\rho_c \, O_c'/O_c \right)$ where prime indicates differentiation.

$L_o = S_c + i P_c \Rightarrow S_c - B + i P_c$ COMMENTS ABOUT COMPLEX NATURE OF L_o
Real ⇒ shift
Img ⇒ total width

eqn. III 4.4 (p. 271)   using eqn. IV 2.6 (p. 274)

*** WHY DO WE HAVE ONE LO MATRIX BUT TWO LEVELS

$$\Longrightarrow \mathbf{L}_o = \begin{pmatrix} S - B + iP & 0 \\ 0 & S' - B' + iP' \end{pmatrix} = \begin{pmatrix} L_o & 0 \\ 0 & L_o' \end{pmatrix}$$

Matrix In Channels

Note: possible confusion in notation for Lo (Both a matrix, and a matrix element.)

$$B^{1/2} = \begin{pmatrix} P^{1/2} & 0 \\ 0 & P'^{1/2} \end{pmatrix}$$

[Reminder.
<u>N.B.</u> We are calculating all these quantities for a given energy E_B(CM)
S, B, P are NUMERICAL QUANTITIES IN AZURE]

To Calculate W^J

First

$$R L_0 = \begin{pmatrix} \frac{\gamma^2}{E_1-E} & \frac{\gamma\gamma'}{E_1-E} \\ \frac{\gamma'\gamma}{E_1-E} & \frac{\gamma'^2}{E_1-E} \end{pmatrix} \begin{pmatrix} L_0 & 0 \\ 0 & L_0' \end{pmatrix} = \begin{pmatrix} \frac{\gamma^2 L_0}{E_1-E} & \frac{\gamma\gamma' L_0'}{E_1-E} \\ \frac{\gamma'\gamma L_0}{E_1-E} & \frac{\gamma'^2 L_0'}{E_1-E} \end{pmatrix}$$

$$1 - R L_0 = \begin{pmatrix} 1-\frac{\gamma^2 L_0}{E_1-E} & -\frac{\gamma\gamma' L_0'}{E_1-E} \\ -\frac{\gamma'\gamma L_0}{E_1-E} & 1-\frac{\gamma'^2 L_0'}{E_1-E} \end{pmatrix}$$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

IS THIS CORRECT FOR COMPLEX MATRICES ??

YES !!

$$(1-RL_0)^{-1} = \left[\left(1-\frac{\gamma^2 L_0}{E_1-E}\right)\left(1-\frac{\gamma'^2 L_0'}{E_1-E}\right) - \left(\frac{\gamma\gamma' L_0'}{E_1-E}\right)\left(\frac{\gamma'\gamma L_0}{E_1-E}\right)\right]^{-1} \begin{pmatrix} 1-\frac{\gamma'^2 L_0'}{E_1-E} & \frac{\gamma\gamma' L_0'}{E_1-E} \\ \frac{\gamma'\gamma L_0}{E_1-E} & 1-\frac{\gamma^2 L_0}{E_1-E} \end{pmatrix}$$

$$(1-RL_0)^{-1} = \frac{1}{1-\frac{L_0\gamma^2}{\Delta E}-\frac{L_0'\gamma'^2}{\Delta E}} \begin{pmatrix} 1-\frac{\gamma'^2 L_0'}{\Delta E} & \frac{\gamma\gamma' L_0'}{\Delta E} \\ \frac{\gamma'\gamma L_0}{\Delta E} & 1-\frac{\gamma^2 L_0}{\Delta E} \end{pmatrix}$$

$$(1-RL_0)^{-1} R = \frac{\Delta E}{\underbrace{E_1-E-L_0\gamma^2-L_0'\gamma'^2}} \begin{pmatrix} 1-\frac{\gamma'^2 L_0'}{\Delta E} & \frac{\gamma\gamma' L_0'}{\Delta E} \\ \frac{\gamma'\gamma L_0}{\Delta E} & 1-\frac{\gamma^2 L_0}{\Delta E} \end{pmatrix} \begin{pmatrix} \frac{\gamma^2}{\Delta E} & \frac{\gamma\gamma'}{\Delta E} \\ \frac{\gamma'\gamma}{\Delta E} & \frac{\gamma'^2}{\Delta E} \end{pmatrix}$$

$$= \frac{\Delta E}{(E_1-E)-(S-B+iP)\gamma^2-(S'-B+iP')\gamma'^2}$$

DERIVE THIS

$$\boxed{= \frac{\Delta E}{(E_1-E) - \underbrace{\gamma^2(S-B)-\gamma'^2(S'-B)}_{\text{Total Shift}} - i\underbrace{(P\gamma^2+P'\gamma'^2)}_{\text{Total Width}}} \times \text{ETC}}$$

EXERCISE FOR STUDENT

To Continue

$$(1-RL)^{-1}R = \frac{\Delta E}{\text{DENOM}}\begin{pmatrix} \frac{\gamma^2}{\Delta E} - \cancel{\frac{\gamma^2\gamma'^2 L_0'}{\Delta E^2}} + \cancel{\frac{\gamma^2\gamma'^2 L_0'}{\Delta E^2}} & \frac{\gamma\gamma'}{\Delta E} - \cancel{\frac{\gamma\gamma'^3 L_0'}{\Delta E^2}} + \cancel{\frac{\gamma\gamma'^3 L_0'}{\Delta E^2}} \\ \cancel{\frac{\gamma^3\gamma' L_0}{\Delta E^2}} + \frac{\gamma\gamma'}{\Delta E} - \cancel{\frac{\gamma^3\gamma' L_0}{\Delta E^2}} & \cancel{\frac{\gamma^2\gamma'^2}{\Delta E^2}} + \frac{\gamma'^2}{\Delta E} - \cancel{\frac{\gamma^2\gamma'^2 L_0}{\Delta E^2}} \end{pmatrix}$$

$$(1-RL)^{-1}R = \frac{\cancel{\Delta E}}{\text{DENOM}}\begin{pmatrix} \frac{\gamma^2}{\cancel{\Delta E}} & \frac{\gamma\gamma'}{\cancel{\Delta E}} \\ \frac{\gamma'\gamma}{\cancel{\Delta E}} & \frac{\gamma'^2}{\cancel{\Delta E}} \end{pmatrix}$$

ALMOST
JUST THE NUMERATOR TERMS WE WANT!!!
(c.f. B-W formula)

$$\boxed{W = 1 + \frac{1}{\text{DENOM}}\begin{pmatrix} P^{1/2} & 0 \\ 0 & P'^{1/2} \end{pmatrix}\begin{pmatrix} \gamma^2 & \gamma\gamma' \\ \gamma\gamma' & \gamma'^2 \end{pmatrix}\begin{pmatrix} P^{1/2} & 0 \\ 0 & P'^{1/2} \end{pmatrix}\cdot 2i}$$

<u>Finally</u>

$$U = \Omega W \Omega \quad \text{where } \Omega = \begin{pmatrix} e^{-i(\omega-\varphi)} & 0 \\ 0 & e^{-i(\omega'-\varphi')} \end{pmatrix}$$

MAJOR EXERCISE FOR STUDENT

Calculate (algebraically) the physical elements of the 2x2 U-matrix for the case of

2 - LEVELS + 2- CHANNELS

NOTE:- W ⇛ MATRIX IN CHANNELS

COMMENT REACTION PATH WAYS